1. For each of the following statements, circle **T** if the statement is always true and **F** if it can be false. **To receive credit, you must give a brief justification for your answer.**

[2] (a) If $\vec{u}$ and $\vec{v}$ are orthogonal vectors in $\mathbb{R}^3$, then $\|\vec{u}+\vec{v}\| = \|\vec{u}\| + \|\vec{v}\|$.

Solution: False. For example, $\vec{u} = [1,0,0]$ and $\vec{v} = [0,1,0]$ are orthogonal, but $\|\vec{u}+\vec{v}\| = \|[1,1,0]\| = \sqrt{2}$ and $\|\vec{u}\| + \|\vec{v}\| = 1+1 = 2$.

[2] (b) If $\vec{u}$ is orthogonal to both $\vec{v}$ and $\vec{w}$, then $\vec{u}$ is orthogonal to $2\vec{v}+3\vec{w}$.

Solution: True, since $\vec{u}\cdot(2\vec{v}+3\vec{w}) = 2\vec{u}\cdot\vec{v} + 3\vec{u}\cdot\vec{w} = 2(0)+3(0) = 0$.

[2] (c) If none of vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^3$ is a scalar multiple of one of the others, then the set $S = \{\vec{u}, \vec{v}, \vec{w}\}$ is linearly independent.

Solution: False. Consider the vectors $[1,0,0]$, $[0,1,0]$ and $[1,1,0]$.

[2] (d) Every system of 3 linear equations in 4 unknowns has infinitely many solutions.

Solution: False. For example, if $x+y+w+z=0$ and $x+y+w+z=1$ are two of the linear equations, then the system has no solutions.

[2] (e) Let $\vec{u} = [3,6]$ and $\vec{v} = [2,4]$. Then $\text{span}(\vec{u}, \vec{v}) = \mathbb{R}^2$.

Solution: False, since they are parallel and therefore span a line.

[2] (f) Let A be a 2×2 matrix. If $B = A^2$, then $AB = BA$.

Solution: True, since $AB = A(A^2) = A^3$ and $BA = (A^2)A = A^3$.

[2] (g) A matrix of the form $\begin{bmatrix} a & 1 \\ -1 & a \end{bmatrix}$ is invertible, for all real numbers a.

Solution: True. Its determinant is $a^2+1 \geq 1 > 0$, so it is invertible.

[2] (h) The set $S = \{[x,y,z] \in \mathbb{R}^3 \mid y = x^2\}$ is a subspace of $\mathbb{R}^3$.

Solution: False. For example, $[1,1,0]$ is in S (since $1^2 = 1$), but $2[1,1,0] = [2,2,0]$ is not in S (since $2^2 \neq 2$).

2. Let $\vec{u} = [2, -1, 3]$ and $\vec{v} = [3, 2, 1]$.

[3] (a) Find the angle between $\vec{u}$ and $\vec{v}$.

Solution: We have

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{7}{\sqrt{14}\sqrt{14}} = \frac{7}{14} = \frac{1}{2}$$

and so $\theta = 60$ degrees.

[2] (b) Compute the projection $\mathrm{proj}_{\vec{u}}(\vec{v})$ of $\vec{v}$ onto $\vec{u}$.

Solution:

$$\mathrm{proj}_{\vec{u}}(\vec{v}) = \frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\,\vec{u} = \frac{7}{14}\,[2, -1, 3] = [1, -1/2, 3/2].$$

[4] 3. Let V be the plane in $\mathbb{R}^3$ given by $2x + y - z = 0$, and let W be the line in $\mathbb{R}^3$ through the origin in the direction of $[1, 0, 1]$. Find vectors $\vec{v} \in V$ and $\vec{w} \in W$ such that $\vec{v} + \vec{w} = [1, 1, 1]$. (Hint: you know $\vec{w} = t[1, 0, 1]$ for some scalar t.)

Solution: Let $\vec{w} = t[1, 0, 1]$. Then $\vec{v} = [1, 1, 1] - t[1, 0, 1] = [1-t, 1, 1-t]$. Since $\vec{v}$ is in the plane, we must have $2(1-t)+1-(1-t) = 0$. That is, $-t+2 = 0$, and so $t = 2$. So $\vec{w} = 2[1, 0, 1] = [2, 0, 2]$ and $\vec{v} = [1, 1, 1] - [2, 0, 2] = [-1, 1, -1]$.

[4] 4. (a) Find a normal vector for the equation of a plane that contains the two lines $\vec{x} = \vec{p} + s\vec{u}$ and $\vec{x} = \vec{p} + t\vec{v}$, where

$$\vec{p} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \quad \vec{u} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \text{and} \quad \vec{v} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}.$$

Solution: Since the plane contains the two lines, it goes through the point $\vec{p}$ and has $\vec{u}$ and $\vec{v}$ as direction vectors. (They are not parallel.)

We take the cross product of the direction vectors to find a normal vector:

$$\vec{u} \times \vec{v} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}.$$

[3] (b) Find a general equation for the plane in part (a).

Solution: Using (a), we know the equation is of the form $x - 3y + 4z = d$. We plug in the point $(1, 0, 2)$ to find that $d = 9$, so the equation is

$$x - 3y + 4z = 9.$$

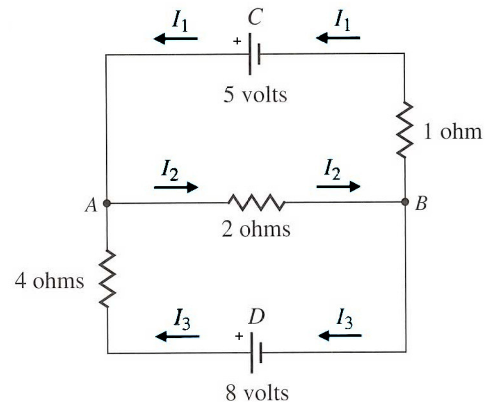
[4] 5. Consider a code that uses code words in $\mathbb{Z}_5^6$ and has check vector $\vec{c} = [2, 1, 2, 1, 2, 1]$. Find the digit d in $\mathbb{Z}_5$ that makes $\vec{v} = [3, 1, 2, 3, 2, d]$ into a valid code word.

Solution: We compute

$$\vec{c} \cdot \vec{v} = 2(3) + 1(1) + 2(2) + 1(3) + 2(2) + 1(d) = 6 + 1 + 4 + 3 + 4 + d = 18 + d = 3 + d$$

in $\mathbb{Z}_5$. To make this 0, we choose $d = 2$.

[5] 6. Determine the currents in this electrical network.



Solution: Kirchhoff's current law at node A gives:

$$I_1 - I_2 + I_3 = 0.$$

At node B, you get the same equation.
Kirchhoff's voltage law for the bottom loop gives:

$$2I_2 + 4I_3 = 8.$$

And for the top loop we get:

$$I_1 + 2I_2 = 5.$$

So we need to solve the system whose augmented matrix is:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 2 & 4 & 8 \\ 1 & 2 & 0 & 5 \end{array}\right]$$

This row reduces to

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

which has solution

$$\begin{aligned} I_3 &= 1 \\ I_2 &= 4 - 2I_3 = 2 \\ I_1 &= I_2 - I_3 = 1 \end{aligned}$$

[4] 7. Let A, B and X be invertible $n \times n$ matrices. Solve the following matrix equation for X in terms of A and B. Show your steps, and simplify your answer.

$$A^T X + (X^T A)^T = B$$

Solution:

$$\begin{aligned} & A^T X + (X^T A)^T = B \\ \implies & A^T X + A^T X = B \\ \implies & 2A^T X = B \\ \implies & X = \frac{1}{2}(A^T)^{-1} B \end{aligned}$$

8. Let $A = \begin{bmatrix} 2 & 1 & 3 \\ 4 & 2 & 2 \\ -2 & 2 & 3 \end{bmatrix}$.

[4] (a) Determine whether A is invertible, and if so, find A^{-1}, showing your work.

Solution: Row reducing $[A|I]$ leads to

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1/12 & 1/8 & -1/6 \\ 0 & 1 & 0 & -2/3 & 1/2 & 1/3 \\ 0 & 0 & 1 & 1/2 & -1/4 & 0 \end{array}\right]$$

and so A is invertible with inverse

$$\begin{bmatrix} 1/12 & 1/8 & -1/6 \\ -2/3 & 1/2 & 1/3 \\ 1/2 & -1/4 & 0 \end{bmatrix}$$

[3] (b) Using your answer to part (a), determine whether the system $A\vec{x} = \vec{0}$ has a unique solution.

Solution: By the fundamental theorem of invertible matrices, it does have a unique solution since A is invertible.

Alternatively, by the above work, we see that there will be no free variables.

9. Let

$$A = \begin{bmatrix} 1 & 3 & 0 & 5 & 0 & 4 \\ 2 & 6 & 1 & 8 & 0 & 5 \\ 2 & 6 & 2 & 6 & 1 & 9 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 3 & 0 & 5 & 0 & 4 \\ 0 & 0 & 1 & -2 & 0 & -3 \\ 0 & 0 & 0 & 0 & 1 & 7 \end{bmatrix}.$$

Given that R is the reduced row-echelon form of A, compute each of the following. **Explain briefly.**

[1] (a) rank(A).

Solution: The rank is 3, since the row echelon form R has three non-zero rows.

[2] (b) A basis for row(A).

Solution: A basis for the row space of A is given by the non-zero rows of R:

$$[1, 3, 0, 5, 0, 4], \quad [0, 0, 1, -2, 0, -3] \quad \text{and} \quad [0, 0, 0, 0, 1, 7]$$

[2] (c) A basis for col(A).

Solution: A basis for the column space of A is given by the columns of A that correspond to columns of R with leading 1's:

$$\begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

[3] (d) A basis for null(A).

Solution: From the matrix R we get the equations

$$x_1 + 3x_2 + 5x_4 + 4x_6 = 0, \quad x_3 - 2x_4 - 3x_6 = 0 \quad \text{and} \quad x_5 + 7x_6 = 0$$

The variables $x_2 = r$, $x_4 = s$ and $x_6 = t$ are free, and the equations above give formulas for x_1, x_3 and x_5, so we find the general solution to be

$$\vec{x} = r \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -5 \\ 0 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ 0 \\ 3 \\ 0 \\ -7 \\ 1 \end{bmatrix},$$

so a basis for the null space is given by the three vectors shown.

10. Let A be a square matrix.

[2] (a) Assume that A is invertible. Show that A^TA is invertible by finding a formula for $(A^TA)^{-1}$ in terms of A^{-1}.

Solution: Recall that, if A is invertible, then $(A^T)^{-1} = (A^{-1})^T$. Also, if A and B are both invertible of the same size, then $(AB)^{-1} = B^{-1}A^{-1}$. Thus, $(A^TA)^{-1} = A^{-1}(A^{-1})^T$ exists.

[2] (b) Now assume that A^TA is invertible. Show that A is invertible by showing that $(A^TA)^{-1}A^T$ is an inverse to A.

Solution: Since A is square, it is enough to show that $(A^TA)^{-1}A^T$ is a one-sided inverse to A, and this is easy to check:

$$((A^TA)^{-1}A^T)A = (A^TA)^{-1}(A^TA) = I.$$

Note that it is *not* valid to write $(A^TA)^{-1} = A^{-1}(A^T)^{-1}$ until we show that A is invertible.

[5] 11. Are the vectors $\vec{u}_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}$, $\vec{u}_3 = \begin{bmatrix} 3 \\ 8 \\ 5 \end{bmatrix}$, and $\vec{u}_4 = \begin{bmatrix} -1 \\ 0 \\ 4 \end{bmatrix}$ linearly dependent? If so, find scalars c_1, c_2, c_3 and c_4 such that $c_1\vec{u}_1 + c_2\vec{u}_2 + c_3\vec{u}_3 + c_4\vec{u}_4 = \vec{0}$.

Solution: A collection of four vectors in $\mathbb{R}^3$ is always linearly dependent. To find the dependency relationship, we have to solve the system

$$\begin{bmatrix} 1 & 0 & 3 & -1 \\ 2 & 2 & 8 & 0 \\ 1 & 3 & 5 & 4 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Row reduction of

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 2 & 2 & 8 & 0 & 0 \\ 1 & 3 & 5 & 4 & 0 \end{array}\right]$$

leads to

$$\left[\begin{array}{cccc|c} 1 & 0 & 3 & -1 & 0 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right]$$

which has general solution

$$c_1 = -5t,\ c_2 = -3t,\ c_3 = 2t,\ c_4 = t.$$

Picking $t = 1$ gives a specific relationship

$$-5\vec{u}_1 - 3\vec{u}_2 + 2\vec{u}_3 + \vec{u}_4 = \vec{0}.$$