1. For each of the following statements, circle **T** if the statement is always true and **F** if it can be false. **Give a brief justification for your answer.**

[2] (a) Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be nonzero vectors in $\mathbb{R}^3$. If $\vec{u}$ and $\vec{v}$ are each orthogonal to $\vec{w}$, then $2\vec{u} - 3\vec{v}$ is orthogonal to $\vec{w}$.

**Solution:** True. We are given $\vec{u} \cdot \vec{w} = 0 = \vec{v} \cdot \vec{w}$, so $(2\vec{u} - 3\vec{v}) \cdot \vec{w} = 2(\vec{u} \cdot \vec{w}) - 3(\vec{v} \cdot \vec{w}) = 2(0) - 3(0) = 0$ and thus $2\vec{u} - 3\vec{v}$ is orthogonal to $\vec{w}$.

[2] (b) The planes $3x - 2y + z = 2$ and $-9x + 6y - 3z = 4$ are parallel.

**Solution:** True. The planes have normal vectors $[3, -2, 1]$ and $[-9, 6, -3]$, which are parallel, so the planes are parallel.

[2] (c) The matrix $\begin{bmatrix} 3 & 6 \\ 1 & 2 \end{bmatrix}$ has rank 2.

**Solution:** False. Subtracting 1/3 of row 1 from row 2 gives $\begin{bmatrix} 3 & 6 \\ 0 & 0 \end{bmatrix}$, which is in row echelon form, so the given matrix has rank 1.

[2] (d) Let $A$ denote the coefficient matrix of a system of 4 linear equations in 3 unknowns. Then this system has a unique solution.

**Solution:** False. For example, if $A$ is the zero matrix and the system is $A\vec{x} = \vec{0}$, then any $\vec{x}$ in $\mathbb{R}^3$ is a solution to the system.

[2] (e) Let $A$ be a $2 \times 3$ matrix. Then the column vectors of $A$ are linearly dependent.

**Solution:** True. Since the homogeneous system $A\vec{x} = \vec{0}$ has 2 equations and 3 unknowns, there is a non-trivial solution, which means that the column vectors of $A$ are linearly dependent.

[2] (f) Let $A$ and $B$ be $2 \times 2$ matrices. Then $AB = BA$.

**Solution:** False. Almost any two random $2 \times 2$ matrices will serve as a counterexample, e.g. $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

[2] (g) The matrix $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is invertible.

**Solution:** True. Its determinant is $1 \cdot 4 - 2 \cdot 3 = -2 \neq 0$, so it is invertible.

[2] (h) The set $S = \{[x, y] \in \mathbb{R}^2 \mid xy = 0\}$ is a subspace of $\mathbb{R}^2$.

**Solution:** False. $[1, 0]$ and $[0, 1]$ are in $S$, but $[1, 0] + [0, 1] = [1, 1]$ is not in $S$.

[2] 2. Suppose that $\vec{u}, \vec{v}, \vec{w} \in \mathbb{R}^4$ are such that $\vec{u} \cdot \vec{v} = -2$, $\vec{u} \cdot \vec{w} = 2$, $\|\vec{v}\| = 3$, and $\vec{v} \cdot \vec{w} = 1$. Compute $(2\vec{u} - \vec{v}) \cdot (3\vec{v} + \vec{w})$.

**Solution:** We have $(2\vec{u}-\vec{v})\cdot(3\vec{v}+\vec{w}) = 6\vec{u}\cdot\vec{v}+2\vec{u}\cdot\vec{w}-3\vec{v}\cdot\vec{v}-\vec{v}\cdot\vec{w} = 6(-2)+2(2)-3(3^2)-1 = -36$.

[2] 3. (a) Find the unit vector in the same direction as $\vec{x} = [2, 1, 2]$.

**Solution:** This will be the vector $\frac{1}{\|\vec{x}\|}\vec{x}$, so we calculate $\|\vec{x}\| = \sqrt{2^2 + 1^2 + 2^2} = \sqrt{9} = 3$ to find that the unit vector in the same direction as $\vec{x}$ is $\frac{1}{3}\vec{x} = [\frac{2}{3}, \frac{1}{3}, \frac{2}{3}]$.

[2] (b) Find the cosine of the angle $\theta$ between $\vec{x} = [2, 1, 2]$ and $\vec{y} = [3, -1, 2]$.

**Solution:**

$$\cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{\|\vec{x}\| \, \|\vec{y}\|} = \frac{[2, 1, 2] \cdot [3, -1, 2]}{\sqrt{2^2 + 1^2 + 2^2}\sqrt{3^2 + (-1)^2 + 2^2}} = \frac{9}{3\sqrt{14}} = \frac{3}{\sqrt{14}}$$

[4] 4. Let $Q = (2, 3)$, $P = (-1, 7)$, and $\vec{d} = [1, 2]$. Find the distance from the point $Q$ to the line through $P$ in the direction of $\vec{d}$.

**Solution:** This will be the length of the projection of $\overrightarrow{PQ}$ onto a normal vector for the line. Since $\vec{n} = [-2, 1]$ is orthogonal to $\vec{d}$, $\vec{n}$ is a normal vector for any line in the direction of $\vec{d}$. Thus we calculate the projection of $\overrightarrow{PQ} = [2, 3] - [-1, 7] = [3, -4]$ along $\vec{n}$ and find its length. The projection of $\overrightarrow{PQ}$ along $\vec{n}$ is the scalar multiple $t\vec{n}$ where $\overrightarrow{PQ} - t\vec{n}$ is orthogonal to $\vec{n}$, so we require $\overrightarrow{PQ} \cdot \vec{n} = t\vec{n} \cdot \vec{n}$; that is, $[3, -4] \cdot [-2, 1] = t[-2, 1] \cdot [-2, 1]$. Thus $t = \frac{-10}{5} = -2$, so we need to calculate $\| - 2\vec{n}\| = 2\|\vec{n}\| = 2\sqrt{5}$.

[3] 5. (a) Find a normal vector for the plane in $\mathbb{R}^3$ that passes through $P = (1, 2, 1)$, $Q = (-1, 1, 0)$, and $R = (2, 1, 3)$.

**Solution:** The vector $\vec{n} = \overrightarrow{RP} \times \overrightarrow{RQ}$ will be a normal vector for the plane. We have $\overrightarrow{RP} = [-1, 1, -2]$ and $\overrightarrow{RQ} = [-3, 0, -3]$, so we compute

$$\begin{array}{rr} -1 & -3 \\ \hline 1 & 0 \\ -2 & -3 \\ -1 & -3 \\ 1 & 0 \end{array} \qquad \begin{array}{l} (1)(-3) - (-2)(0) = -3 \\ (-2)(-3) - (-1)(-3) = 3 \\ (-1)(0) - (1)(-3) = 3 \end{array}$$

Thus $\vec{n} = [-3, 3, 3]$ is a normal vector for the plane (as is any nonzero scalar multiple of $\vec{n}$).

[3] (b) Find a general equation for the plane in (a).

**Solution:** $X = [x, y, z]$ is on the plane if and only if $(X - P) \cdot \vec{n} = 0$; that is, $[x - 1, y - 2, z - 1] \cdot [-3, 3, 3] = 0$. We find that $-3x + 3 + 3y - 6 + 3z - 3 = 0$, or $-3x + 3y + 3z = 6$. We could also write $-x + y + z = 2$. Verify that $P$, $Q$, and $R$ are on this plane. $-1 + 2 + 1 = 2$, so $P$ lies on this plane. $-(-1) + 1 + 0 = 2$, so $Q$ lies on this plane. Finally, $-(2) + 1 + 3 = 2$ so $R$ lies on this plane.

[3] 6. Find a vector equation for the line of intersection of the planes $2x-2y+2z=4$ and $2x-y+3z=1$.

**Solution:** We solve the system of two linear equations $2x-2y+2z=4$, $2x-y+3z=1$ in three unknowns $x, y, z$ using row reduction

$$\left[\begin{array}{ccc|c} 2 & -2 & 2 & 4 \\ 2 & -1 & 3 & 1 \end{array}\right] \xrightarrow{\frac{1}{2}R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 2 & -1 & 3 & 1 \end{array}\right]$$

$$\xrightarrow{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|c} 1 & 0 & 2 & -1 \\ 0 & 1 & 1 & -3 \end{array}\right]$$

This gives solutions $x=-1-2t$, $y=-3-t$ and $z=t$. So a vector form of the equation for this line is $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ -3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$. Other solutions are also possible.

7. Recall that the Universal Product Code (UPC) uses code words in $\mathbb{Z}_{10}^{12}$ and has check vector $c = [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1]$.

[3] (a) Is the vector $[0, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6]$ a valid UPC? Explain.

**Solution:** The vector $[0, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6]$ is a valid UPC if its dot product with $c$ (calculated in $\mathbb{Z}_{10}$) is 0. We calculate

$$\begin{aligned} &[0, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6] \cdot [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \\ &= (0)(3) + (1)(1) + (3)(3) + (2)(1) + (4)(3) + (3)(1) \\ &\qquad + (5)(3) + (4)(1) + (6)(3) + (5)(1) + (7)(3) + (6)(1) \\ &= 1 + 9 + 2 + 12 + 3 + 15 + 4 + 18 + 5 + 21 + 6 \equiv 2 + 2 + 3 + 5 + 4 + 8 + 5 + 1 + 6 \\ &= 36 \equiv 6 \neq 0 \mod 10 \end{aligned}$$

so $[0, 1, 3, 2, 4, 3, 5, 4, 6, 5, 7, 6]$ is not a valid UPC.

[3] (b) Find the missing digit $y$ in the UPC $[0, 1, 2, 1, 3, 9, y, 5, 0, 7, 3, 4]$.

**Solution:** Since $[0, 1, 2, 1, 3, 9, y, 5, 0, 7, 3, 4]$ is a valid UPC code, its dot product with $c$ (calculated in $\mathbb{Z}_{10}$ is 0. We calculate

$$\begin{aligned} &[0, 1, 2, 1, 3, 9, y, 5, 0, 7, 3, 4] \cdot [3, 1, 3, 1, 3, 1, 3, 1, 3, 1, 3, 1] \\ &= (0)(3) + (1)(1) + (2)(3) + (1)(1) + (3)(3) + (9)(1) \\ &\qquad + (y)(3) + (5)(1) + (0)(3) + (7)(1) + (3)(3) + (4)(1) \\ &= 1 + 6 + 1 + 9 + 9 + 3y + 5 + 7 + 9 + 4 \equiv 3y + 1 \mod 10 \end{aligned}$$

so $y \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$ must be such that $3y + 1$ is a multiple of 10. Thus $y = 3$.

[5] 8. Determine whether the vector $\vec{v} = \begin{bmatrix} 12 \\ 3 \\ 4 \end{bmatrix}$ is in the span of the vectors $\vec{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$, $\vec{u}_2 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\vec{u}_3 = \begin{bmatrix} 6 \\ 3 \\ 4 \end{bmatrix}$. If it is, find a way to express $\vec{v}$ as a linear combination of $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$.

**Solution:** Row reduction of

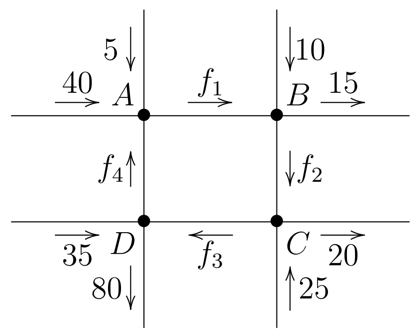
$$\left[\begin{array}{ccc|c} 2 & 1 & 6 & 12 \\ 1 & 2 & 3 & 3 \\ 3 & 1 & 4 & 4 \end{array}\right]$$

leads to

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 3 \end{array}\right]$$

and so it follows that $\vec{v} = -2\vec{u}_1 - 2\vec{u}_2 + 3\vec{u}_3$. So $\vec{v}$ is in the span of $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$.

9. Consider the pictured network of water pipes, where the flows are measured in litres/minute. In this question, flows can be positive or negative.




[2] (a) Set up a system of linear equations for the possible flows $f_1$, $f_2$, $f_3$ and $f_4$.

**Solution:**

$$\begin{aligned} A:&\ 45 + f_4 = f_1 \\ B:&\ 10 + f_1 = 15 + f_2 \\ C:&\ f_2 + 25 = 20 + f_3 \\ D:&\ 35 + f_3 = f_4 + 80 \end{aligned}$$

or

$$\begin{array}{ccccccccc} f_1 & & & & & - & f_4 & = & 45 \\ f_1 & - & f_2 & & & & & = & 5 \\ & & f_2 & - & f_3 & & & = & -5 \\ & & & & f_3 & - & f_4 & = & 45 \end{array}$$

[2] (b) Solve the system of equations.

**Solution:** One can do the usual method of row reduction, or the following ad hoc method: $f_3 = 45 + f_4$, so $f_2 = -5 + 45 + f_4 = 40 + f_4$, and $f_1 = 45 + f_4$, with $f_4$ a free variable.

10. Let $A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 5 & 0 \\ -1 & -2 & 3 \end{bmatrix}$.

[4] (a) Determine whether $A$ is invertible, and if so, find $A^{-1}$, showing your work.

**Solution:**

$$A^{-1} = \begin{bmatrix} 15 & -2 & 10 \\ -6 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix}$$

[3] (b) Solve the system $A\vec{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ for $\vec{x}$.

**Solution:** We have that

$$\vec{x} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 15 & -2 & 10 \\ -6 & 1 & -4 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ -3 \\ 1 \end{bmatrix}$$

It also works to solve the system directly, but this way is more efficient since we already know $A^{-1}$.

11. Let

$$A = \begin{bmatrix} 2 & 4 & 2 & 0 & 2 \\ 3 & 5 & 3 & -2 & 7 \\ 4 & 7 & 4 & -2 & 8 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 1 & 0 & 1 & -4 & 9 \\ 0 & 1 & 0 & 2 & -4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given that $R$ is the reduced row-echelon form of $A$, compute each of the following. Explain briefly.

[1] (a) rank($A$).

**Solution:** The rank is 2, since the row echelon form $R$ has two non-zero rows.

[2] (b) A basis for row($A$).

**Solution:** A basis for the row space of $A$ is given by the non-zero rows of $R$:

$$[1, 0, 1, -4, 9] \quad \text{and} \quad [0, 1, 0, 2, -4]$$

[2] (c) A basis for col($A$).

**Solution:** A basis for the column space of $A$ is given by the columns of $A$ that correspond to columns of $R$ with leading 1's:

$$\begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}$$

[3] (d) A basis for null($A$).

**Solution:** From the matrix $R$ we get the equations

$$x_1 + x_3 - 4x_4 + 9x_5 = 0 \quad \text{and} \quad x_2 + 2x_4 - 4x_5 = 0$$

The variables $x_3 = r$, $x_4 = s$ and $x_5 = t$ are free, and the equations above give formulas for $x_1$ and $x_2$, so we find the general solution to be

$$\begin{bmatrix} -r + 4s - 9t \\ -2s + 4t \\ r \\ s \\ t \end{bmatrix} = r \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 4 \\ -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -9 \\ 4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

so a basis for the null space is given by the three vectors shown.

[4] 12. Let $A$, $B$ and $X$ be invertible $n \times n$ matrices. Solve the following matrix equation for $X$ in terms of $A$ and $B$. Show your steps, and simplify your answer.

$$(BX + A)^T = A + A^T + B^T$$

**Solution:**

$$\begin{aligned} (BX + A)^T &= A + A^T + B^T \\ \implies BX + A &= (A + A^T + B^T)^T = A^T + A + B \\ \implies BX &= A^T + A + B - A = A^T + B \\ \implies X &= B^{-1}(A^T + B) = B^{-1}A^T + I \end{aligned}$$