THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

MATHEMATICS 1600B MIDTERM EXAMINATION
31 January 2013 7:00–8:30 PM

1. For each of the following statements, circle **T** if the statement is always true and **F** if it can be false. Give a one-sentence justification for your answer.

   (a) (2 marks) If two non-zero vectors are parallel, then the angle between them is 0 degrees.

   Solution: This is FALSE. The angle can also be 180 degrees, for example if $\mathbf{v}$ is non-zero and $\mathbf{w} = -\mathbf{v}$.

   (b) (2 marks) For any two vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^3$, $\|\mathbf{u} \times \mathbf{v}\| \leq \|\mathbf{u}\|\|\mathbf{v}\|$.

   Solution: This is TRUE. We have $\|\mathbf{u} \times \mathbf{v}\| = \|\mathbf{u}\|\|\mathbf{v}\| \sin\theta \leq \|\mathbf{u}\|\|\mathbf{v}\|$ because for the angle θ between $\mathbf{u}$ and $\mathbf{v}$ one has $\sin\theta \leq 1$.

   (c) (2 marks) Any vector in the plane is a linear combination of the standard unit vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.

   Solution: This is TRUE because

$$\begin{bmatrix} x \\ y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \end{bmatrix} = x\,\mathbf{e}_1 + y\,\mathbf{e}_2.$$

2. (2 marks) Let $\mathbf{u} = [3, 2, 1]$ and $\mathbf{v} = [1, 3, 2]$ be in $(\mathbb{Z}_5)^3$. Find all scalars b in $\mathbb{Z}_5$ such that $(\mathbf{u} + b\mathbf{v}) \cdot (b\mathbf{u} + \mathbf{v}) = 1$. (Read again the last instruction on the front page!)

   Solution:

$$\begin{aligned} 1 &= (\mathbf{u} + b\mathbf{v}) \cdot (b\mathbf{u} + \mathbf{v}) = b\,\mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + b^2\,\mathbf{v} \cdot \mathbf{u} + b\,\mathbf{v} \cdot \mathbf{v} \\ &= b\,(\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v}) + (b^2 + 1)\,\mathbf{u} \cdot \mathbf{v} = 28b + (b^2 + 1) \cdot 11 = b^2 + 3b + 1. \end{aligned}$$

   Hence $b^2 + 3b = b(b + 3) = 0$, which means $b = 0$ or $b = -3 = 2$.

3. Let $\mathbf{v} = [\sqrt{3}, 0, 1]$ and $\mathbf{w} = [-1, \sqrt{2}, 1]$.

(a) (2 marks) Compute the length of $\mathbf{v}+\mathbf{w}$.

Solution:

$$\begin{aligned}\mathbf{v}+\mathbf{w} &= \begin{bmatrix}\sqrt{3}-1\\ \sqrt{2}\\ 2\end{bmatrix},\\ \left\|\mathbf{v}+\mathbf{w}\right\|^2 &= (\sqrt{3}-1)^2+2+4 = 10-2\sqrt{3},\\ \left\|\mathbf{v}+\mathbf{w}\right\| &= \sqrt{10-2\sqrt{3}}.\end{aligned}$$

(b) (1 mark) Find a unit vector pointing in the same direction as $\mathbf{w}$.

Solution:

$$\mathbf{u} = \frac{1}{\|\mathbf{w}\|}\,\mathbf{w} = \frac{1}{2}\,\mathbf{w} = \begin{bmatrix}-1/2\\ 1/\sqrt{2}\\ 1/2\end{bmatrix}.$$

(c) (2 marks) Find all unit vectors $\mathbf{x}$ in the xy-plane that make an angle of 30 degrees with $\mathbf{v}$.

Solution: Let $\mathbf{x} = [x, y, 0]$ be a unit vector in the xy-plane. Since the angle between $\mathbf{x}$ and $\mathbf{v}$ equals 30 degrees, we have

$$\cos 30^\circ = \frac{1}{2}\sqrt{3} = \frac{\mathbf{x}\cdot\mathbf{v}}{\|\mathbf{x}\|\,\|\mathbf{v}\|} = \frac{1}{2}\sqrt{3}\,x$$

or $x = 1$. Since $\mathbf{x}$ is a unit vector, this implies $y = 0$, so that $\mathbf{x} = [1, 0, 0]$ is the only solution.

4. (2 marks) Give an example of a valid UPC code $\mathbf{x}$ with the following two properties:

- If one reads the UPC code backwards, one gets again a valid UPC code.
- The 'backwards' code is not the same as the original code. (That is, codes of the form $\mathbf{x} = [0\,1\,2\,3\,4\,5\,5\,4\,3\,2\,1\,0]$ are not permitted.)

(Recall that the check vector for UPC is $\mathbf{c} = [3\,1\,3\,1\,3\,1\,3\,1\,3\,1\,3\,1]$.)

Solution: Let $\mathbf{x} = [x_1\, x_2 \ldots x_{11}\, x_{12}]$. The first condition means that

$$\begin{aligned} 3x_1 + \ \ x_2 + 3x_3 + \ \ x_4 + \cdots + 3x_{11} + \ \ x_{12} &= 0, \\ x_1 + 3x_2 + \ \ x_3 + 3x_4 + \cdots + \ \ x_{11} + 3x_{12} &= 0. \end{aligned}$$

Subtracting three times the second equation from the first we get

$$-\,8x_2 \qquad -\,8x_4 - \cdots - \qquad -\,8x_{12} = 0,$$

that is,

$$2x_2 + \qquad 2x_4 + \cdots + \qquad +\,2x_{12} = 0.$$

If we set $x_2 = x_4 = \cdots = x_{12} = 0$, the second equation becomes

$$x_1 + \qquad x_3 + \qquad \cdots + \ \ x_{11} \qquad = 0.$$

Choosing $x_1 = 1$ and $x_3 = -1 = 9$ and $x_5 = x_7 = x_9 = x_{11} = 0$ therefore is one possible solution to the problem,

$$\mathbf{x} = [1\,0\,9\,0\,0\,0\,0\,0\,0\,0\,0\,0].$$

5. Let $A(2,2,4)$, $B(-1,0,5)$, $C(3,4,3)$ be three points in $\mathbb{R}^3$.

(a) (2 marks) Write a vector form of the equation of the plane containing the triangle ΔABC.

Solution: The vectors $\mathbf{u} = \overrightarrow{AB} = B - A = [-3,-2,1]$ and $\mathbf{v} = \overrightarrow{AC} = C - A = [1,2,-1]$ are parallel to the plane, but not on the same line. Hence a vector form is

$$\mathbf{x} = A + s\,\mathbf{u} + t\,\mathbf{v} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix} + s \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}.$$

(b) (2 marks) Give the normal form of the equation of the plane containing the triangle ΔABC.

Solution: A normal vector of the plane is

$$\mathbf{n} = \mathbf{u} \times \mathbf{v} = \begin{bmatrix} -3 \\ -2 \\ 1 \end{bmatrix} \times \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \\ -4 \end{bmatrix},$$

hence a normal form of the equation is

$$\mathbf{n} \cdot \mathbf{x} = \mathbf{n} \cdot A = -20.$$

(c) (2 marks) What is the area of the triangle ΔABC?

Solution: The area is given by

$$\frac{1}{2}\|\mathbf{u}\times\mathbf{v}\| = \frac{1}{2}\|\mathbf{n}\| = \frac{\sqrt{20}}{2} = \sqrt{5}.$$

(d) (1 mark) Find a point in the plane from part (a) that lies *inside* (not including the edges) of the triangle ΔABC.

Solution: Using the vector form of the plane from part (a), we may choose any point $\mathbf{x} = A + s\mathbf{u} + t\mathbf{v}$ with s and t small enough so that the $s\mathbf{u} + t\mathbf{v}$ lies in the triangle determined by $\mathbf{u}$ and $\mathbf{v}$ (precisely, any s, t with $s + t < 1$ will do). For example, choosing $s = t = 1/3$, we see that $\mathbf{x} = (4/3, 2, 4)$ lies inside the triangle. (This particular point

$$Q = \frac{1}{3}(A + B + C) = (4/3, 2, 4).$$

is known as the barycentre or centroid. See the "exploration" on page 32 of the book.)

6. Consider the system of linear equations

$$\begin{aligned} x + 3y + 2z \qquad &= 1 \\ -z + 2w &= 0 \\ 3x + 5y - z \qquad &= 5 \\ 2x + 2y - 2z - 2w &= 4 \end{aligned}$$

(a) (1 mark) Write down the augmented matrix of this linear system.

Solution:

$$\left[\begin{array}{rrrr|r} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & -1 & 2 & 0 \\ 3 & 5 & -1 & 0 & 5 \\ 2 & 2 & -2 & -2 & 4 \end{array}\right]$$

(b) (3 marks) Compute the reduced row-echelon form of the augmented matrix above. Indicate all elementary row operations that you are performing.

Solution:

$$\xrightarrow[R_4-2R_1]{R_3-3R_1} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & -4 & -7 & 0 & 2 \\ 0 & -4 & -6 & -2 & 2 \end{array}\right] \xrightarrow{R_4-R_3} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & -4 & -7 & 0 & 2 \\ 0 & 0 & 1 & -2 & 0 \end{array}\right]$$

$$\xrightarrow{R_2\leftrightarrow R_3\leftrightarrow R_4} \left[\begin{array}{cccc|c} 1 & 3 & 2 & 0 & 1 \\ 0 & -4 & -7 & 0 & 2 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & -1 & 2 & 0 \end{array}\right] \xrightarrow[\substack{R_2+7R_3 \\ R_4+R_3}]{R_1-2R_3} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 1 \\ 0 & -4 & 0 & -14 & 2 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{R_2/(-4)} \left[\begin{array}{cccc|c} 1 & 3 & 0 & 4 & 1 \\ 0 & 1 & 0 & 7/2 & -1/2 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \xrightarrow{R_1-3R_2} \left[\begin{array}{cccc|c} 1 & 0 & 0 & -13/2 & 5/2 \\ 0 & 1 & 0 & 7/2 & -1/2 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

(c) (2 marks) Use the result of the previous part to find all solutions of the linear system.

Solution: We have one parameter $t = w$ and

$$\begin{aligned} x - 13/2\, w &= 5/2, \\ y + 7/2\, w &= -1/2, \\ z - 2\, w &= 0, \end{aligned}$$

hence

$$\begin{aligned} x &= 5/2 + 13/2\, t, \\ y &= -1/2 - 7/2\, t, \\ z &= 2t, \\ w &= t. \end{aligned}$$

7. Let $\mathcal{P}$ be the plane through the point $A(2, -1, -1)$ with normal vector $\mathbf{n} = [2, 2, -1]$.

(a) (2 marks) Compute parametric equations of the plane $\mathcal{P}$.

Solution: We need two vectors parallel to $\mathcal{P}$, but not lying on a common line. We choose $\mathbf{u} = [1, 0, 2]$ and $\mathbf{v} = [0, 1, 2]$. Hence a vector equation for $\mathcal{P}$ is

$\mathbf{x} = A + s\,\mathbf{u} + t\,\mathbf{v}$, and parametric equations are

$$\begin{aligned} x &= 2 + s, \\ y &= -1 + t, \\ z &= -1 + 2s + 2t. \end{aligned}$$

(b) (2 marks) Compute the distance from $\mathcal{P}$ to the origin.

Solution: The distance d from $\mathcal{P}$ to the origin is the length of the projection of $\overrightarrow{AO} = [-2, 1, 1]$ onto $\mathbf{n}$. We have

$$\operatorname{proj}_{\mathbf{n}} \overrightarrow{AO} = \frac{\overrightarrow{AO} \cdot \mathbf{n}}{\mathbf{n} \cdot \mathbf{n}}\,\mathbf{n} = \frac{-3}{9}\,\mathbf{n} = -\frac{1}{3}\mathbf{n}.$$

Since $\mathbf{n}$ has length 3, the distance is $d = 1$.

(c) (1 mark) Find a plane $\mathcal{P}'$ that is parallel to $\mathcal{P}$ and has distance 2 from $\mathcal{P}$.

Solution: The normal vector $\mathbf{n}$ has length 3, so that the point $B = A \pm 2/3\,\mathbf{n}$ has distance 2 from $\mathcal{P}$. This is also the distance between $\mathcal{P}$ and the plane $\mathcal{P}'$ through B parallel to $\mathcal{P}$. Its normal form is

$$\mathbf{x} \cdot \mathbf{n} = B \cdot \mathbf{n} = (A \pm 2/3\,\mathbf{n}) \cdot \mathbf{n} = A \cdot \mathbf{n} \pm 2/3\,\mathbf{n} \cdot \mathbf{n} = 3 \pm 6.$$

Hence one solution is $\mathbf{x} \cdot \mathbf{n} = 9$ and the other one is $\mathbf{x} \cdot \mathbf{n} = -3$.