1. For each of the following statements, circle **T** if the statement is always true and **F** if it can be false. Give a one-sentence justification for your answer.

[2] (a) The set $S = \{[x, y] \in \mathbb{R}^2 \mid x + y \geqslant -1\}$ is a subspace of $\mathbb{R}^2$.

Solution: False. $[1, 1]$ is in S, but $(-1)[1, 1] = [-1, -1]$ is not in S.

[2] (b) Let A be a square matrix. If the column vectors of A are linearly independent, then the row vectors of A are linearly independent too.

Solution: True, by the fundamental theorem of invertible matrices.

[2] (c) Let M be a 3×2 matrix. Then the column vectors of M are linearly dependent.

Solution: False. For example, the columns of

$$M = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

are linearly independent.

[2] (d) If A and B are both symmetric $n \times n$ matrices and c and d are any scalars, then $cA + dB$ is also symmetric.

Solution: True. Symmetric means $A^T = A$ and $B^T = B$. Using properties of transpose,

$$(cA + dB)^T = (cA)^T + (dB)^T = c(A^T) + d(B^T) = cA + dB$$

so $cA + dB$ is symmetric too.

[2] (e) The dimension of $\text{span}\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ is 3.

Solution: False. The third vector is the sum of the first two, so a basis for this span has two elements. Thus the dimension is two, not three.

[3] 2. Let A, B and X be invertible $n \times n$ matrices. Solve the following matrix equation for X in terms of A and B. Show your steps, and simplify your answer.

$$A(X^T + B) = (B + A)B$$

Solution:

$$\begin{aligned} A(X^T + B) &= (B + A)B \\ \implies X^T + B &= A^{-1}(B + A)B = A^{-1}B^2 + B \\ \implies X^T &= A^{-1}B^2 \\ \implies X &= (A^{-1}B^2)^T = (B^2)^T(A^{-1})^T = (B^T)^2(A^T)^{-1} \end{aligned}$$

Any of the three forms on the last line is correct.

[3] 3. Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 2 \\ 2 & 2 & 3 \end{bmatrix}$$

Solution: We row reduce $[A \mid I]$:

$$\left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 2 & 2 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-2R_1]{R_2-R_1} \left[\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1+R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right] \xrightarrow{R_1+2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & 2 \\ 0 & -1 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -2 & 0 & 1 \end{array}\right]$$

$$\xrightarrow[-1\cdot R_3]{-1\cdot R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -4 & 1 & 2 \\ 0 & 1 & 0 & 1 & -1 & 0 \\ 0 & 0 & 1 & 2 & 0 & -1 \end{array}\right] \quad \checkmark$$

We find

$$A^{-1} = \begin{bmatrix} -4 & 1 & 2 \\ 1 & -1 & 0 \\ 2 & 0 & -1 \end{bmatrix}$$

Don't forget to check your answer by multiplying!

4. Let

$$A = \begin{bmatrix} 0 & 1 & -4 & -1 & 1 \\ 0 & 3 & -12 & 0 & 9 \\ 0 & 2 & -8 & 2 & 10 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 0 & 1 & -4 & 0 & 3 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Given that R is the reduced row-echelon form of A, compute each of the following. Explain briefly.

[1] (a) rank(A).

Solution: The rank is 2, since the row echelon form R has two non-zero rows.

[1] (b) nullity(A).

Solution: $\text{rank}(A) + \text{nullity}(A) = 5$, so $\text{nullity}(A) = 5 - 2 = 3$.

[2] (c) A basis for row(A).

Solution: A basis for the row space of A is given by the non-zero rows of R:

$$[0, 1, -4, 0, 3] \quad \text{and} \quad [0, 0, 0, 1, 2]$$

[2] (d) A basis for col(A).

Solution: A basis for the column space of A is given by the columns of A that correspond to columns of R with leading 1's:

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix}$$

[3] (e) A basis for null(A).

Solution: From the matrix R we get the equations

$$x_2 - 4x_3 + 3x_5 = 0 \quad \text{and} \quad x_4 + 2x_5 = 0$$

The variables $x_1 = r$, $x_3 = s$ and $x_5 = t$ are free, and the equations above give formulas for x_2 and x_4, so we find the general solution to be

$$\begin{bmatrix} r \\ 4s-3t \\ s \\ -2t \\ t \end{bmatrix} = r\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ 4 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ -3 \\ 0 \\ -2 \\ 1 \end{bmatrix}$$

so a basis for the null space is given by the three vectors shown.

[3] 5. Let A be a 2×4 matrix. What are the possible values of the rank and nullity of A? Explain.

Solution: Since A has only 2 rows, the rank can be at most 2. So the possible values for the rank are 0, 1 and 2. We also know that $\text{rank}(A) + \text{nullity}(A) = 4$, so the nullity of A can be 4, 3 or 2.

6. Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be a linear transformation such that

$$T(\vec{e}_1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{and} \quad T(\vec{e}_2) = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$

[2] (a) Find the standard matrix $[T]$ of T.

Solution: $[T] = [\,T(\vec{e}_1) \mid T(\vec{e}_2)\,] = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}$

[3] (b) Determine whether T is invertible. If so, find the standard matrix of T^{-1}.

Solution: The determinant of $[T]$ is $(1)(4) - (2)(3) = -2$, so the matrix $[T]$ is invertible. Therefore, the linear transformation T is invertible. The standard matrix of T^{-1} is the inverse of $[T]$, which is

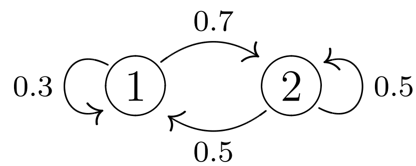
$$\frac{1}{-2}\begin{bmatrix} 4 & -3 \\ -2 & 1 \end{bmatrix}$$

[2] (c) Find the standard matrix of $T^2 = T \circ T$.

Solution:

$$[T^2] = [T]^2 = \begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 7 & 15 \\ 10 & 22 \end{bmatrix}$$

7. Consider the following Markov process:



[2] (a) Write down the transition matrix P for this Markov process.

Solution: The ij entry is the probability of going from state j to state i:

$$P = \begin{bmatrix} 0.3 & 0.5 \\ 0.7 & 0.5 \end{bmatrix}$$

[3] (b) Find a steady state probability vector for P.

Solution: We solve the system $(I - P)\vec{x} = \vec{0}$.

$$[\, I - P \mid 0 \,] = \left[\begin{array}{rr|r} 0.7 & -0.5 & 0 \\ -0.7 & 0.5 & 0 \end{array}\right] \to \left[\begin{array}{rr|r} 0.7 & -0.5 & 0 \\ 0.0 & 0.0 & 0 \end{array}\right]$$

so $0.7x_1 = 0.5x_2$. $x_2 = t$ is free, and $x_1 = \frac{5}{7}x_2 = \frac{5}{7}t$, so we get $\begin{bmatrix} \frac{5}{7}t \\ t \end{bmatrix}$. For this to be a probability vector, we need $\frac{5}{7}t + t = \frac{12}{7}t = 1$, so $t = 7/12$ and $x_1 = 5/12$. So the solution is $\begin{bmatrix} \frac{5}{12} \\ \frac{7}{12} \end{bmatrix}$.