THE UNIVERSITY OF WESTERN ONTARIO
DEPARTMENT OF MATHEMATICS

MATHEMATICS 1600B MIDTERM EXAMINATION
07 March 2013 7:00–8:30 PM

1. For each of the following statements, circle **T** if the statement is always true and **F** if it can be false. Give a one-sentence justification for your answer.

   (a) (2 marks) If $z^2 + w^2 = 0$, where z and w are complex numbers, then it must be that both $z = 0$ and $w = 0$.

   Solution: This is FALSE.
   If $z^2 + w^2 = 0$, we can have $z = iw$ or $z = -iw$. In particular, choosing $z = 1$ and $w = i$ provides a counter-example.

   (b) (2 marks) If the vectors $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly dependent, then $\mathbf{u}$ is a linear combination of $\mathbf{v}$ and $\mathbf{w}$.

   Solution: This is FALSE.
   For a counter-example, consider $\mathbf{u} = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v} = \begin{bmatrix}0\\1\end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix}0\\0\end{bmatrix}$ in $\mathbb{R}^2$.

2. (a) (1 mark) Give an example of a matrix A such that the columns of A are linearly independent, while the rows of A are linearly dependent.

   Solution: Let $A = \begin{bmatrix}1 & 0\\0 & 1\\0 & 0\end{bmatrix}$. The matrix has rank 2. Hence the two column vectors are linearly independent and the three row vectors are linearly dependent.

   (b) (1 mark) Give an example of two square matrices A and B such that $(A + B)^2 \neq A^2 + 2AB + B^2$.

   Solution: We always have $(A+B)^2 = A^2+AB+BA+B^2$. Hence the requirement that $(A + B)^2 \neq A^2 + 2AB + B^2$ is equivalent to $AB \neq BA$.
   Let $A = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ and $B = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$. Then $AB = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix}$ and $BA = \begin{bmatrix}0 & 1\\1 & 1\end{bmatrix}$ are different from each other. (Note: “Almost any” choice of A and B will give a counter-example.)

3. (2 marks) Find the complex number z satisfying $(1+i)z + (1-i)^2 = 6 - 2i$. (Express your answer in the form $z = a + bi$.)

Solution:

$$z = \frac{6-2i-(1-i)^2}{1+i} = \frac{6-2i-(-2i)}{1+i} = \frac{6(1-i)}{(1+i)(1-i)} = \frac{6-6i}{2} = 3-3i.$$

4. (2 marks) Assume that X, A and B are square matrices of the same size, and that A and B are invertible. Solve the following matrix equation for X.

$$B^{-1}A^{-1}(X+A) = BA^{-1} + B^{-1}$$

Solution: We multiply the matrix AB from the left to both sides of the equation to cancel $B^{-1}A^{-1}$:

$$ABB^{-1}A^{-1}(X+A) = AB(BA^{-1} + B^{-1}).$$

Since $ABB^{-1}A^{-1} = I$, we get

$$X + A = AB^2A^{-1} + A.$$

$$X = AB^2A^{-1}.$$

5. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ be a matrix with coefficients in $\mathbb{Z}_3$.

   (a) (2 marks) Compute $A^2 + A$.

Solution:

$$A^2 + A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \text{ or}$$

$$A^2 + A = A(A+I) = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

   (b) (2 marks) Find all 2×2 matrices X (with coefficients in $\mathbb{Z}_3$) such that $AX = 0$.

Solution: Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then $AX = 0$ if and only if

$$\begin{aligned} a + 2c &= 0 \\ 2a + c &= 0 \\ b + 2d &= 0 \\ 2b + d &= 0 \end{aligned}$$

The matrix $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 1 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ (with coefficients in $\mathbb{Z}_3$). Thus $a + 2c = b + 2d = 0$, or $a = c$ and $b = d$. (Note that in the matrix we omitted the rightmost column because it consists entirely of zeros. Also, by observing that the pair a and c satisfies exactly the same linear equations as the pair b and d, one could have solved this more quickly.)

As a result, the matrices X such that $AX = 0$ are of the form $\begin{bmatrix} a & b \\ a & b \end{bmatrix}$ for $a = 0, 1, 2$ and $b = 0, 1, 2$ in $\mathbb{Z}_3$. There are nine of them.

6. Let $A = \begin{bmatrix} -2 & 1 & 2 \\ -1 & 2 & 2 \\ -2 & 4 & 3 \end{bmatrix}$.

   (a) (3 marks) Calculate A^{-1}.

Solution:

$$\left[\begin{array}{ccc|ccc} -2 & 1 & 2 & 1 & 0 & 0 \\ -1 & 2 & 2 & 0 & 1 & 0 \\ -2 & 4 & 3 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{-R_2} \left[\begin{array}{ccc|ccc} -2 & 1 & 2 & 1 & 0 & 0 \\ 1 & -2 & -2 & 0 & -1 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & -1 & 0 \\ -2 & 1 & 2 & 1 & 0 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2+2R_1} \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & -1 & 0 \\ 0 & -3 & -2 & 1 & -2 & 0 \\ 0 & 3 & 1 & -1 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R_3+R_2} \left[\begin{array}{ccc|ccc} 1 & -2 & -2 & 0 & -1 & 0 \\ 0 & -3 & -2 & 1 & -2 & 0 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right] \xrightarrow[R_2-2R_3]{R_1-2R_3} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 3 & -2 \\ 0 & -3 & 0 & 1 & 2 & -2 \\ 0 & 0 & -1 & 0 & -2 & 1 \end{array}\right]$$

$$\xrightarrow[-R_3]{R_2/(-3)} \left[\begin{array}{ccc|ccc} 1 & -2 & 0 & 0 & 3 & -2 \\ 0 & 1 & 0 & -1/3 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right] \xrightarrow{R_1+2R_2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -2/3 & 5/3 & -2/3 \\ 0 & 1 & 0 & -1/3 & -2/3 & 2/3 \\ 0 & 0 & 1 & 0 & 2 & -1 \end{array}\right]$$

$$\text{Hence } A^{-1} = \begin{bmatrix} -2/3 & 5/3 & -2/3 \\ -1/3 & -2/3 & 2/3 \\ 0 & 2 & -1 \end{bmatrix}.$$

(b) (1 mark) Find z in the system of linear equations below. (You may wish to use your answer from the previous part.)

$$\begin{array}{rrrcl} -2x & +y & +2z & = & 5 \\ -x & +2y & +2z & = & -3 \\ -2x & +4y & +3z & = & 1 \end{array}$$

Solution: The system is the same as

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}.$$

Multipling by A^{-1} from the left, we get

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix}$$

By taking the third row of A^{-1}, we get in particular

$$z = \begin{bmatrix} 0 & 2 & -1 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 1 \end{bmatrix} = -7.$$

(Note that one doesn't need to compute the values of x and y.)

7. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 2 \\ 3 \end{bmatrix}$, $\mathbf{v}_4 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

(a) (3 marks) Determine whether $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is a spanning set for $\mathbb{R}^3$.

Solution: We form the matrix $A = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 2 & 4 \\ 2 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$, so that

$$span(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4) = \text{row}(A).$$

The matrix A row reduces to $\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\\0&0&0\end{bmatrix}$. Hence $\{\begin{bmatrix}1&0&0\end{bmatrix},\begin{bmatrix}0&1&0\end{bmatrix},\begin{bmatrix}0&0&1\end{bmatrix}\}$ is a basis for $span(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)$. It follows that $span(\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4)=\mathbb{R}^3$.
Alternative argument: If you found a different basis, then it still has three elements. So the span is a 3-dimensional subspace of $\mathbb{R}^3$. Since we know all subspaces of $\mathbb{R}^3$, this can only be the whole space.

(b) (1 mark) Is $\{\mathbf{v}_1,\mathbf{v}_2,\mathbf{v}_3,\mathbf{v}_4\}$ linearly independent? Explain.

Solution: No, the vectors are not linearly independent, since a set of linearly independent vectors in $\mathbb{R}^n$ can contain at most n vectors. Here $n=3$, but we have 4 vectors.

There is an alternative solution:

Solution: We need to find all solutions for the homogeneous linear system corresponding to the matrix $\begin{bmatrix}1&1&2&0\\1&2&2&1\\2&4&3&0\end{bmatrix}$, which row reduces to $\begin{bmatrix}1&0&0&-5\\0&1&0&1\\0&0&1&2\end{bmatrix}$.
It follows that there is a non-zero relation between the vectors, namely

$$5\mathbf{v}_1-\mathbf{v}_2-2\mathbf{v}_3+\mathbf{v}_4=\mathbf{0}.$$

8. Let

$$A=\begin{bmatrix}1&-2&1&4&-1&9\\1&-2&2&7&-1&11\\-1&2&-1&-4&2&-12\\1&-2&-1&-2&-2&8\end{bmatrix}\quad\text{and}\quad R=\begin{bmatrix}1&-2&0&1&0&4\\0&0&1&3&0&2\\0&0&0&0&1&-3\\0&0&0&0&0&0\end{bmatrix}.$$

Given that R is the reduced row-echelon form of A, find the following, explaining your work.

(a) (1 mark) A basis for row(A)

Solution: The non-zero rows in R form a basis for row(A). Hence

$$\{\begin{bmatrix}1&-2&0&1&0&4\end{bmatrix},\begin{bmatrix}0&0&1&3&0&2\end{bmatrix},\begin{bmatrix}0&0&0&0&1&-3\end{bmatrix}\}$$

is a basis for row(A).

(b) (2 marks) A basis for col(A)

Solution: The columns in A corresponding to the columns containing leading 1's in R form a basis for col(A). Hence

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 2 \\ -2 \end{bmatrix} \right\}$$

is a basis for col(A).

(c) (2 marks) A basis for null(A)

Solution: By solving the system of (homogeneous) linear equations corresponding to the matrix R, we find that

$$\begin{aligned} x_1 &= 2s - t - 4r \\ x_2 &= s \\ x_3 &= -3t - 2r \\ x_4 &= t \\ x_5 &= 3r \\ x_6 &= r. \end{aligned}$$

Putting the solution into vector form, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = s \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + r \begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 3 \\ 1 \end{bmatrix}.$$

The vectors in the vector form of the solution forms a basis for null(A), i.e.,

$$\left\{ \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -2 \\ 0 \\ 3 \\ 1 \end{bmatrix} \right\}$$

is a basis for null(A).

9. For a complex number $z = a + bi$, consider the real 2×2 matrix $A_z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. For example, in the case $z = i$ one has $A_i = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.

(a) (1 mark) If $z = e^{(3\pi/4)i}$, find A_z.

Solution:

$$e^{(3\pi/4)i} = \cos(3\pi/4) + i\sin(3\pi/4) = -\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i.$$

$$\text{Hence } A_{e^{(3\pi/4)i}} = \begin{bmatrix} -\frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \end{bmatrix}.$$

(b) (2 marks) Compute $A_i^{100} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}^{100}$.

Solution: We can compute that $A_i^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ and $A_i^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$$\text{Hence } A_i^{100} = (A_i^4)^{25} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{25} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

(c) (2 marks) Compute $(A_z)^{-1}$ for $z \neq 0$.

Solution: $A_z = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$. If $z \neq 0$, then $det(A_z) = a^2 + b^2 \neq 0$ and A_z is invertible, with

$$(A_z)^{-1} = \frac{1}{\det(A_z)} \begin{bmatrix} a & b \\ -b & a \end{bmatrix} = \begin{bmatrix} \frac{a}{a^2+b^2} & \frac{b}{a^2+b^2} \\ -\frac{b}{a^2+b^2} & \frac{a}{a^2+b^2} \end{bmatrix}.$$

(Note that this is $A_{1/z}$. This is not a coincidence; we might talk about this in class sometime.)