Math 1600B Lecture 13, Section 2, 3 Feb 2014

Announcements:

Read Sections 3.1 and 3.2 for next class. Work through recommended homework questions.

Quiz 4 is this week, and will focus on the material in Section 2.3 (linear (in)dependence), 2.4 (networks) and part of 3.1 (what we cover today).

Office hour: today, 1:30-2:30, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Lecture 12:

We covered network analysis and electrical networks in Section 2.4. Since the material won't be used today, I won't summarize it. I didn't quite finish Exercise 2.4.20, so I leave it as an exercise to solve the system I derived on the board:

$$\begin{aligned} I_1 + 2I_2 &= 5 \\ 2I_2 + 4I_3 &= 8 \\ I_1 - I_2 + I_3 &= 0 \end{aligned}$$

We aren't covering Section 2.5.

New material: Section 3.1: Matrix Operations

(Lots of definitions, but no tricky concepts.)

Definition: A **matrix** is a rectangular array of numbers called the **entries**. The entries are usually real (from $\mathbb{R}$), but may also be complex (from $\mathbb{C}$).

Examples:

$$A = \begin{bmatrix} 1 & -3/2 & \pi \\ \sqrt{2} & 2.3 & 0 \end{bmatrix}$$

2×3

$$\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}$$

1×3

row matrix or **row vector**

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

3×1

column matrix or **column vector**

The entry in the ith row and jth column of A is usually written a_{ij} or sometimes A_{ij}. For example,

$$A_{11} = 1, \quad A_{23} = 0, \quad A_{32} \text{ doesn't make sense.}$$

Definition: An $m \times n$ matrix A is **square** if $m = n$. The **diagonal entries** are $a_{11}, a_{22}, \ldots$. If A is square and the nondiagonal entries are all zero, then A is called a **diagonal matrix**.

$$\begin{bmatrix} 1 & -3/2 & \pi \\ \sqrt{2} & 2.3 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

not square or diagonal — square — diagonal — diagonal

Definition: A diagonal matrix with all diagonal entries equal is called a **scalar matrix**. A scalar matrix with diagonal entries all equal to 1 is an **identity matrix**.

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \qquad I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

scalar — identity matrix — scalar

Note: Identity $\implies$ scalar $\implies$ diagonal $\implies$ square.

Now we're going to mimick a lot of what we did when we first introduced vectors.

Definition: Two matrices are **equal** if they have the same size and their corresponding entries are equal.

$$\begin{bmatrix} 1 & -3/2 \\ \sqrt{2} & 0 \end{bmatrix} \quad \begin{bmatrix} \cos 0 & -1.5 \\ \sqrt{2} & \sin 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 & 2 & 3 & 4 \end{bmatrix} \quad \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

The first two above are equal, but no other two are equal. We distinguish row matrices from column matrices!

Matrix addition and scalar multiplication

Our first two operations are just like for vectors:

Definition: If A and B are both $m \times n$ matrices, then their **sum** $A + B$ is the $m \times n$ matrix obtained by adding the corresponding entries of A and B. Using the notation $A = [a_{ij}]$ and $B = [b_{ij}]$, we write

$$A + B = [a_{ij} + b_{ij}] \qquad \text{or} \qquad (A + B)_{ij} = a_{ij} + b_{ij}.$$

Examples:

$$\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} + \begin{bmatrix} 0 & -1 & 2 \\ \pi & 0 & -6 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 5 \\ 4 + \pi & 5 & 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & -1 \end{bmatrix} \quad \text{is not defined}$$

Definition: If A is an $m \times n$ matrix and c is a scalar, then the **scalar multiple** cA is the $m \times n$ matrix obtained by multiplying each entry by c. We write $cA = [c\,a_{ij}]$ or $(cA)_{ij} = c\,a_{ij}$.

Example:

$$3 \begin{bmatrix} 0 & -1 & 2 \\ \pi & 0 & -6 \end{bmatrix} = \begin{bmatrix} 0 & -3 & 6 \\ 3\pi & 0 & -18 \end{bmatrix}$$

Definition: As expected, $-A$ means $(-1)A$ and $A - B$ means $A + (-B)$.

The $m \times n$ **zero matrix** has all entries 0 and is denoted O or $O_{m \times n}$. Of course, $A + O = A$.

So we have the real number 0, the zero vector $\vec{0}$ (or $\mathbf{0}$ in the text) and the zero matrix O.

Matrix multiplication

This is *unlike* anything we have seen for vectors.

Definition: If A is $m \times n$ and B is $n \times r$, then the **product** $C = AB$ is the $m \times r$ matrix whose i, j entry is

$$c_{ij} = a_{i1}b_{1j} + a_{i2}b_{2j} + \cdots + a_{in}b_{nj} = \sum_{k=1}^{n} a_{ik}b_{kj}.$$

This is the *dot product* of the ith row of A with the jth column of B.

Note that for this to make sense, the number of columns of A must equal the number of rows of B.

$$\begin{array}{ccccc} A & & B & = & AB \\ m \times n & & n \times r & & m \times r \end{array}$$

This may seem very strange, but it turns out to be useful. We will never use componentwise multiplication, as it is not generally useful.

Examples on board: 2×3 times 3×4, 1×3 times 3×1, 3×1 times 1×3.

One motivation for this definition of matrix multiplication is that it comes up in linear systems.

Example 3.8: Consider the system

$$\begin{aligned} 4x + 2y &= 4 \\ 5x + y &= 8 \\ 6x + 3y &= 6 \end{aligned}$$

The left-hand sides are in fact a matrix product:

$$\begin{bmatrix} 4 & 2 \\ 5 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Every linear system with augmented matrix $[A \mid \vec{b}\,]$ can be written as $A\vec{x} = \vec{b}$.

Note: In general, if A is $m \times n$ and B is a column vector in $\mathbb{R}^n$ ($n \times 1$), then AB is a column vector in $\mathbb{R}^m$ ($m \times 1$). So one thing a matrix A can do is *transform* column vectors into column vectors. This point of view will be important later.

Question: If A is an $m \times n$ matrix and $\vec{e}_1$ is the first standard unit vector in $\mathbb{R}^n$, what is $A\vec{e}_1$?

The answer is an $m \times 1$ column matrix, whose ith entry is the dot product of the ith row of A with the vector $\vec{e}_1$. But $[a_{i1}, a_{i2}, \ldots, a_{in}] \cdot [1, 0, \ldots, 0] = a_{i1}$, the first entry. So this just "picks out" the first column of A. For example,

$$\begin{bmatrix} 4 & 2 \\ 5 & 1 \\ 6 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

More generally, we have:

Theorem 3.1: If A is $m \times n$, $\vec{e}_i$ is the ith $1 \times m$ standard row vector and $\vec{e}_j$ is the jth $n \times 1$ standard column vector, then

$$\vec{e}_i A = \text{the } i\text{th row of } A$$

and

$$A\vec{e}_j = \text{the } j\text{th column of } A.$$

Powers

In general, $A^2 = AA$ doesn't make sense. But if A is $n \times n$ (square), then $A^2 = AA$ does make sense. A^2 is $n \times n$ as well, and so it also makes sense to define the **power**

$$A^k = AA \cdots A \quad \text{with } k \text{ factors.}$$

We write $A^1 = A$ and $A^0 = I_n$.

We will see later that $(AB)C = A(BC)$, so the expression for A^k is unambiguous. And it follows that

$$A^r A^s = A^{r+s} \qquad \text{and} \qquad (A^r)^s = A^{rs}$$

for all nonnegative integers r and s.

Example 3.13 on board: Powers of

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

True/false: Every diagonal matrix is a scalar matrix.

False. But every scalar matrix is diagonal.

True/false: If A is diagonal, then so is A^2.

True. Write $\vec{r}_i$ for the ith row of A and $\vec{c}_j$ for the jth column. The ij entry of A^2 is the dot product $\vec{r}_i \cdot \vec{c}_j$. If $i \neq j$, then the non-zero entry of $\vec{r}_i$ (which is in the ith spot) doesn't line up with the non-zero entry of $\vec{c}_j$ (in the jth spot), so $\vec{r}_i \cdot \vec{c}_j = 0$.

True/false: If A and B are both square, then AB is square.

False. For example, if A is 2×2 and B is 3×3 then AB is not defined. But if A and B are square of the same size, then AB is defined and is also square.

Challenge question (for next class): Is there a nonzero matrix A such that $A^2 = O$?

Next class: We'll cover the properties these operations have, from Section 3.2.