Math 1600B Lecture 2, Section 2, 8 Jan 2014

Announcements:

Read Section 1.2 for next class. Work through homework problems.

Lecture notes (this page) available from course web page. Also look for **announcements** there.

Office hour: today, 2:30-3:30, MC103B. Drop with any questions!

No tutorials this week. There is a quiz in tutorials next week.

Please **read over syllabus**, especially before e-mailing me with questions, as it covers all of the main points.

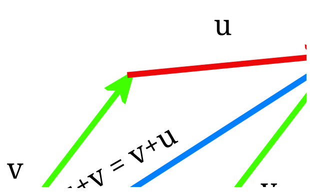
Let me know if the bookstore runs out of **texts or combo packs**.

Review of last lecture:

A vector can be represented by its list of components, e.g. $[1, 2, -1]$ is a vector in $\mathbb{R}^3$.
We write $\mathbb{R}^n$ for the set of all vectors with n real components, e.g. $[1, 2, 3, 4, 5, 6, 7]$ is in $\mathbb{R}^7$.

We also often write vectors as column vectors, e.g. $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Vector addition: $[u_1, \ldots, u_n] + [v_1, \ldots, v_n] := [u_1 + v_1, \ldots, u_n + v_n]$.
E.g. $[3, 2, 1] + [1, 0, -1] = [4, 2, 0]$.



Scalar multiplication: $c[u_1, \ldots, u_n] := [cu_1, \ldots, cu_n]$.

E.g. $2[1, 2, 3, 4, 5] = [2, 4, 6, 8, 10]$.

Zero vector: $\vec{0} := [0, 0, \ldots, 0]$.

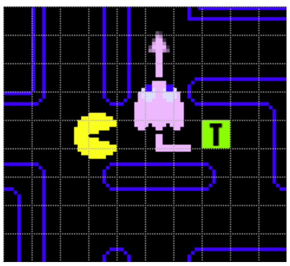
Properties of vector operations: The parallelogram shows geometrically that vector addition is *commutative*: $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.

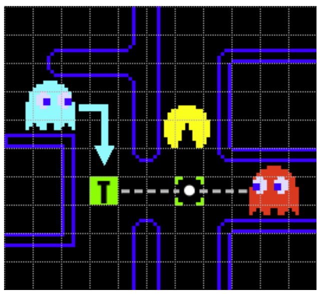
Many other properties that hold for real numbers also hold for vectors: Theorem 1.1. But we'll see differences later.

New material:

An important real-world application:

Pac-Man: Google's version, and How the ghosts move.





Derive an equation for Inky's target on board.

Section 1.1, continued: Linear combinations

Definition: A vector $\vec{v}$ is a **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there are scalars $c_1, c_2, \ldots, c_k$ so that

$$\vec{v} = c_1 \, \vec{v}_1 + \cdots + c_k \, \vec{v}_k .$$

The numbers $c_1, \ldots, c_k$ are called the coefficients. They are not necessarily unique.

Example: Is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$?

Yes, since

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0 \begin{bmatrix} 2 \\ -1 \end{bmatrix} - 2 \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(Check!)}$$

Note: We also have

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = -\frac{1}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \text{(Check!)}$$

and many more possibilities.

We will learn later how to find all solutions.

Example: Is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

No, since any linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$ has a zero as the second component.

Example: Is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ a linear combination of $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$?

Yes. The zero vector is a linear combination of *any* set of vectors, since you can just take $c_1 = c_2 = \cdots = c_k = 0$.

Coordinates

Example: Express $\vec{w}_1 = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$ as a linear combination of $\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

We can solve this by using $\vec{u}$ and $\vec{v}$ to make a new coordinate system in the plane. Use the board to show that $\vec{w}_1 = 2\,\vec{u} + \vec{v}$.

Similarly, show that $\vec{w}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ can be expressed as $\vec{w}_2 = \vec{u} - 2\,\vec{v}$.

Note that in this case the coefficients are unique. In this situation, the coefficients are called the **coordinates** with respect to $\vec{u}$ and $\vec{v}$. So the coordinates of $\vec{w}_1$ with respect to $\vec{u}$ and $\vec{v}$ are 2 and 1, and the coordinates of $\vec{w}_2$ with respect to $\vec{u}$ and $\vec{v}$ are 1 and -2.

Working in a different coordinate system is a powerful tool.

Binary vectors

$\mathbb{Z}_2 := \{0, 1\}$

Multiplication is as usual.

Addition: $0 + 0 = 0,\ 0 + 1 = 1,\ 1 + 0 = 1,\ 1 + 1 = 0$.

$\mathbb{Z}_2^n :=$ vectors with n components in $\mathbb{Z}_2$.

E.g. $[0, 1, 1, 0, 1] \in \mathbb{Z}_2^5$.

$[0, 1, 1] + [1, 1, 0] = [1, 0, 1]$ in $\mathbb{Z}_2^3$.

There are 2^n vectors in $\mathbb{Z}_2^n$.

Ternary vectors

$\mathbb{Z}_3 := \{0, 1, 2\}$

To add and multiply, always take the remainder modulo 3 at the end.

E.g. $2 + 2 = 4 = 1 \cdot 3 + 1$, so $2 + 2 = 1 \pmod 3$.

We write $\pmod 3$ to indicate we are working in $\mathbb{Z}_3$.

Similarly, $1 + 2 = 0 \pmod 3$ and $2 \cdot 2 = 1 \pmod 3$.

$\mathbb{Z}_3^n :=$ vectors with n components in $\mathbb{Z}_3$.

$[0, 1, 2] + [1, 2, 2] = [1, 0, 1]$ in $\mathbb{Z}_3^3$.

There are 3^n vectors in $\mathbb{Z}_3^n$.

Vectors in $\mathbb{Z}_m^n$

$\mathbb{Z}_m := \{0, 1, 2, \ldots, m - 1\}$ with addition and multiplication modulo m.

E.g., in $\mathbb{Z}_{10}$, $\quad 8 \cdot 8 = 64 = 4 \pmod{10}$.

$\mathbb{Z}_m^n :=$ vectors with n components in $\mathbb{Z}_m$.

To find solutions to an equation such as

$$6x = 6 \pmod{8}$$

you can simply try all possible values of x. In this case, 1 and 5 both work, and no other value works.

Note that you can not in general **divide** in $\mathbb{Z}_m$, only add, subtract and multiply. For example, there is no solution to the following equation:

$$2x = 1 \pmod{4}$$

But there is a solution to

$$2x = 1 \pmod{5},$$

namely $x = 3$

Question: In $\mathbb{Z}_5$, what is -2?

Example 1.40 (UPC Codes): The Univeral Product Code (bar code) on a product is a vector in $\mathbb{Z}_{10}^{12}$, such as

$$\vec{u} = [6, 7, 1, 8, 6, 0, 0, 1, 3, 6, 2, 4].$$

6 71860 01362 4

In Section 1.4, we will learn how error detection works for codes like this.