Math 1600B Lecture 27, Section 2, 14 March 2014

Announcements:

Today we (mostly) finish 4.3. **Read** Appendix C and section 4.4 for next class. Work through recommended homework questions.

Tutorials: Quiz 8 covers 4.2 and 4.3, including the parts of Appendix D that we covered.

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Midterm average: 53/70 = 76%

Question: If P is invertible, how do $\det A$ and $\det(P^{-1}AP)$ compare?

They are equal:

$$\begin{aligned}\det(P^{-1}AP) &= \det(P^{-1})\det(A)\det(P) \\ &= \frac{1}{\det(P)}\det(A)\det(P) = \det A.\end{aligned}$$

Partial review of last class: Section 4.3

Definition: If A is $n \times n$, $\det(A - \lambda I)$ will be a degree n polynomial in λ. It is called the **characteristic polynomial** of A, and $\det(A - \lambda I) = 0$ is called the **characteristic equation**.

By the fundamental theorem of invertible matrices, the solutions to the characteristic equation are exactly the eigenvalues.

Finding eigenvalues and eigenspaces: Let A be an $n \times n$ matrix.

1. Compute the characteristic polynomial $\det(A - \lambda I)$.
2. Find the eigenvalues of A by solving the characteristic equation

$\det(A - \lambda I) = 0$.
3. For each eigenvalue λ, find a basis for the eigenspace $E_\lambda = \text{null}(A - \lambda I)$ by solving the system $(A - \lambda I)\vec{x} = \vec{0}$.

So we need to get good at solving polynomial equations. Solutions are called **zeros** or **roots**.

Theorem D.4 (The Fundamental Theorem of Algebra): A polynomial of degree n has at most n distinct roots.

Therefore:

Theorem: An $n \times n$ matrix A has at most n distinct eigenvalues.

Also:

Theorem D.2 (The Factor Theorem): Let f be a polynomial and let a be a constant. Then a is a zero of $f(x)$ (i.e. $f(a) = 0$) if and only if $x - a$ is a factor of $f(x)$ (i.e. $f(x) = (x - a)g(x)$ for some polynomial g).

New material: 4.3 continued

Example 4.18: Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$.

Solution: 1. Last time, we computed the characteristic polynomial:

$$\det(A - \lambda I) = -\lambda^3 + 4\lambda^2 - 5\lambda + 2$$

2. Then we found that

$$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = -(\lambda - 1)^2(\lambda - 2)$$

with roots $\lambda = 1$ and $\lambda = 2$.

3. To find the $\lambda = 1$ eigenspace, we do row reduction:

$$[A - I \mid 0] = \left[\begin{array}{rrr|r} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 2 & -5 & 3 & 0 \end{array}\right] \rightarrow \left[\begin{array}{rrr|r} 1 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We find that $x_3 = t$ is free and $x_1 = x_2 = x_3$, so

$$E_1 = \left\{ \begin{bmatrix} t \\ t \\ t \end{bmatrix} \right\} = \text{span}\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$$

So $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a basis of the eigenspace corresponding to $\lambda = 1$. Check!

Finding a basis for E_2 is similar; see text.

A root a of a polynomial f implies that $f(x) = (x - a)g(x)$. Sometimes, a is also a root of $g(x)$, as we found above. Then $f(x) = (x - a)^2 h(x)$. The largest k such that $(x - a)^k$ is a factor of f is called the **multiplicity** of the root a in f.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

In the previous example, $\lambda = 1$ has algebraic multiplicity 2 and $\lambda = 2$ has algebraic multiplicity 1.

We also define the **geometric multiplicity** of an eigenvalue λ to be the dimension of the corresponding eigenspace. In the previous example, $\lambda = 1$ has geometric multiplicity 1 (and so does $\lambda = 2$).

Example 4.19: Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$. Do partially, on board.

In this case, we find that $\lambda = 0$ has algebraic multiplicity 2 and geometric multiplicity 2.

These multiplicities will be important in Section 4.4.

Theorem 4.15: The eigenvalues of a triangular matrix are the entries on its main diagonal (repeated according to their algebraic multiplicity).

Example: If $A = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ 4 & 5 & 1 \end{bmatrix}$, then

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 2 & 3-\lambda & 0 \\ 4 & 5 & 1-\lambda \end{vmatrix} = (1-\lambda)^2(3-\lambda),$$

so the eigenvalues are $\lambda = 1$ (with algebraic multiplicity 2) and $\lambda = 3$ (with algebraic multiplicity 1).

Question: What are the eigenvalues of a diagonal matrix?

The eigenvalues are the diagonal entries.

Question: What are the eigenvalues of $\begin{bmatrix} 0 & 4 \\ 1 & 0 \end{bmatrix}$?

The characteristic polynomial is

$$\begin{vmatrix} -\lambda & 4 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 4 = (\lambda - 2)(\lambda + 2),$$

so the eigenvalues are 2 and -2. Trick question.

Question: How can we tell whether a matrix A is invertible using eigenvalues?

A is invertible if and only if 0 is not an eigenvalue, because 0 being an eigenvalue is equivalent to $\operatorname{null}(A)$ being non-trivial, which is equivalent to A not being invertible, by the fundamental theorem.

So we can extend the fundamental theorem with two new entries:

Theorem 4.17: Let A be an $n \times n$ matrix. The following are equivalent:
a. A is invertible.
b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.
c. $A\vec{x} = \vec{0}$ has only the trivial (zero) solution.
d. The reduced row echelon form of A is I_n.
f. $\text{rank}(A) = n$
g. $\text{nullity}(A) = 0$
h. The columns of A are linearly independent.
i. The columns of A span $\mathbb{R}^n$.
j. The columns of A are a basis for $\mathbb{R}^n$.
k. The rows of A are linearly independent.
l. The rows of A span $\mathbb{R}^n$.
m. The rows of A are a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A

Next: how to become a Billionaire using the material from this course.