Math 1600B Lecture 28, Section 2, 17 Mar 2014

Announcements:

Today we finish 4.3 and discuss Appendix C. **Read** Section 4.4 for next class. Work through recommended homework questions.

Tutorials: Quiz 8 covers 4.2, 4.3, and the parts of Appendix D that we covered in class.

Help Centers: Monday-Friday 2:30-6:30 in MC 106.

Office hour: Wednesday, 10:30-11:15, MC103B. (Today's hour is cancelled.)

Brief review of last lecture:

The **characteristic polynomial** of a square matrix A is $\det(A - \lambda I)$, which is a polynomial in λ. The roots/zeros of this polynomial are the eigenvalues of A.

A root a of a polynomial f implies that $f(x) = (x-a)g(x)$. Sometimes, a is also a root of $g(x)$. Then $f(x) = (x-a)^2 h(x)$. The largest k such that $(x-a)^k$ is a factor of f is called the **multiplicity** of the root a in f.

In the case of an eigenvalue, we call its multiplicity in the characteristic polynomial the **algebraic multiplicity** of this eigenvalue.

For example, if $\det(A - \lambda I) = -(\lambda - 1)^2 (\lambda - 2)$, then $\lambda = 1$ is an eigenvalue with algebraic multiplicity 2, and $\lambda = 2$ is an eigenvalue with algebraic multiplicity 1.

We also define the **geometric multiplicity** of an eigenvalue λ to be the dimension of the corresponding eigenspace.

Theorem 4.15: The eigenvalues of a triangular matrix are the entries on its main diagonal (repeated according to their algebraic multiplicity).

Theorem 4.17: Let A be an $n \times n$ matrix. The following are equivalent:
a. A is invertible.
b. $A\vec{x} = \vec{b}$ has a unique solution for every $\vec{b} \in \mathbb{R}^n$.
c. $A\vec{x} = \vec{0}$ has only the trivial (zero) solution.
d. The reduced row echelon form of A is I_n.

f. $\mathrm{rank}(A) = n$
g. $\mathrm{nullity}(A) = 0$
h. The columns of A are linearly independent.
i. The columns of A span $\mathbb{R}^n$.
j. The columns of A are a basis for $\mathbb{R}^n$.
k. The rows of A are linearly independent.
l. The rows of A span $\mathbb{R}^n$.
m. The rows of A are a basis for $\mathbb{R}^n$.
n. $\det A \neq 0$
o. 0 is not an eigenvalue of A

New Material: 4.3: Eigenvalues of powers and inverses

Suppose $\vec{x}$ is an eigenvector of A with eigenvalue λ. What can we say about A^2 or A^3? If A is invertible, how about the eigenvalues/vectors of A^{-1}? On board.

We've shown:

Theorem 4.18: If $\vec{x}$ is an eigenvector of A with eigenvalue λ, then $\vec{x}$ is an eigenvector of A^k with eigenvalue λ^k. This holds for each integer $k \geq 0$, and also for $k < 0$ if A is invertible.

In contrast to some other recent results, this one is very useful computationally:

Example 4.21: Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Solution: By finding the eigenspaces of the matrix, we can show that

$$\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = - \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Write $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, $\vec{x} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, $\vec{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Since $\vec{x} = 3\vec{v}_1 + 2\vec{v}_2$ we have

$$\begin{aligned} A^{10}\vec{x} &= A^{10}(3\vec{v}_1 + 2\vec{v}_2) = 3A^{10}\vec{v}_1 + 2A^{10}\vec{v}_2 \\ &= 3(-1)^{10}\vec{v}_1 + 2(2^{10})\vec{v}_2 = \begin{bmatrix} 3 + 2^{11} \\ -3 + 2^{12} \end{bmatrix} \end{aligned}$$

Much faster than repeated matrix multiplication, especially if 10 is replaced with 100.

This raises an interesting question. In the example, the eigenvectors were a basis for $\mathbb{R}^2$, so we could use this method to compute $A^k\vec{x}$ for any $\vec{x}$. However, last class we saw a 3×3 matrix with two one-dimensional eigenspaces, so the eigenvectors didn't span $\mathbb{R}^3$. We will study this further in Section 4.4, but right now we can answer a related question about linear independence.

Theorem: If $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are eigenvectors of A corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_m$, then $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m$ are linearly independent.

Proof in case $m = 2$: If $\vec{v}_1$ and $\vec{v}_2$ are linearly dependent, then $\vec{v}_1 = c\vec{v}_2$ for some c. Therefore

$$A\vec{v}_1 = A\,c\vec{v}_2 = cA\vec{v}_2$$

so

$$\lambda_1\vec{v}_1 = c\lambda_2\vec{v}_2 = \lambda_2\vec{v}_1$$

Since $\vec{v}_1 \neq \vec{0}$, this forces $\lambda_1 = \lambda_2$, a contradiction. $\square$

The general case is very similar; see text.

Appendix C: Complex numbers

Sometimes a polynomial has complex numbers as its roots, so we need to learn a bit about them.

A **complex number** is a number of the form $a + bi$, where a and b are real numbers and i is a symbol such that $i^2 = -1$.

If $z = a + bi$, we call a the **real part** of z, written $\operatorname{Re} z$, and b the **imaginary part** of z, written $\operatorname{Im} z$.

Complex numbers $a + bi$ and $c + di$ are **equal** if $a = c$ and $b = d$.

On board: sketch complex plane and various points.

Addition: $(a + bi) + (c + di) = (a + c) + (b + d)i$, like vector addition.

Multiplication: $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$. (Explain.)

Examples: $(1+2i)+(3+4i)=4+6i$

$$\begin{aligned}(1+2i)(3+4i) &= 1(3+4i)+2i(3+4i) = 3+4i+6i+8i^2 \\ &= (3-8)+10i = -5+10i\end{aligned}$$

$$5(3+4i)=15+20i$$

$$(-1)(c+di)=-c-di$$

The **conjugate** of $z=a+bi$ is $\bar{z}=a-bi$. Reflection in real axis. We'll use this for division of complex numbers in a moment.

Theorem (Properties of conjugates): Let w and z be complex numbers. Then:
1. $\bar{\bar{z}}=z$
2. $\overline{w+z}=\bar{w}+\bar{z}$
3. $\overline{wz}=\bar{w}\bar{z}$ (typo in text) (good exercise)
4. If $z\neq 0$, then $\overline{w/z}=\bar{w}/\bar{z}$ (see below for division)
5. z is **real** if and only if $\bar{z}=z$

The **absolute value** or **modulus** $|z|$ of $z=a+bi$ is

$$|z|=|a+bi|=\sqrt{a^2+b^2}, \quad \text{the distance from the origin.}$$

Note that

$$z\bar{z}=(a+bi)(a-bi)=a^2-abi+abi-b^2i^2=a^2+b^2=|z|^2$$

This means that for $z\neq 0$

$$\frac{z\bar{z}}{|z|^2}=1 \quad \text{so} \quad z^{-1}=\frac{\bar{z}}{|z|^2}$$

This can be used to compute quotients of complex numbers:

$$\frac{w}{z}=\frac{w}{z}\,\frac{\bar{z}}{\bar{z}}=\frac{w\bar{z}}{|z|^2}\,.$$

Example:

$$\frac{-1+2i}{3+4i}=\frac{-1+2i}{3+4i}\,\frac{3-4i}{3-4i}=\frac{5+10i}{3^2+4^2}=\frac{5+10i}{25}=\frac{1}{5}+\frac{2}{5}\,i$$

Theorem (Properties of absolute value): Let w and z be complex numbers.

Then:
1. $|z| = 0$ if and only if $z = 0$.
2. $|\bar{z}| = |z|$
3. $|wz| = |w||z|$ (good exercise!)
4. If $z \neq 0$, then $|w/z| = |w|/|z|$. In particular, $|1/z| = 1/|z|$.
5. $|w + z| \leq |w| + |z|$.

Polar Form

A complex number $z = a + bi$ can also be expressed in **polar coordinates** (r, θ), where $r = |z| \geq 0$ and θ is such that

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta \qquad \text{(sketch)}$$

Then

$$z = r\cos\theta + (r\sin\theta)i = r(\cos\theta + i\sin\theta)$$

To compute θ, note that

$$\tan\theta = \sin\theta / \cos\theta = b/a.$$

But this doesn't pin down θ, since $\tan(\theta + \pi) = \tan\theta$. You must choose θ based on what quadrant z is in. There is a unique correct θ with $-\pi < \theta \leq \pi$, and this is called the **principal argument** of z and is written $\operatorname{Arg} z$ (or $\arg z$).

Examples: If $z = 1 + i$, then $r = |z| = \sqrt{1^2 + 1^2} = \sqrt{2}$. By inspection, $\theta = \pi/4 = 45^\circ$. We also know that $\tan\theta = 1/1 = 1$, which gives $\theta = \pi/4 + k\pi$, and $k = 0$ gives the right quadrant.

We write $\operatorname{Arg} z = \pi/4$ and $z = \sqrt{2}(\cos\pi/4 + i\sin\pi/4)$.

If $w = -1 - i$, then $r = \sqrt{2}$ and by inspection $\theta = -3\pi/4 = -135^\circ$. We *still* have $\tan\theta = -1/-1 = 1$, which gives $\theta = \pi/4 + k\pi$, but now we must take k odd to land in the right quadrant. Taking $k = -1$ gives the principal argument:

$$\operatorname{Arg} w = -3\pi/4 \quad \text{and} \quad w = \sqrt{2}(\cos(-3\pi/4) + i\sin(-3\pi/4)).$$

Multiplication and division in polar form

Let

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \quad \text{and} \quad z_2 = r_2(\cos\theta_2 + i\sin\theta_2).$$

Then

$$\begin{aligned} z_1 z_2 &= r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1 r_2 [(\cos\theta_1 \cos\theta_2 - \sin\theta_1 \sin\theta_2) + i(\sin\theta_1 \cos\theta_2 + \cos\theta_1 \sin\theta_2)] \\ &= r_1 r_2 [\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)] \end{aligned}$$

So

$$|z_1 z_2| = |z_1||z_2| \quad \text{and} \quad \operatorname{Arg}(z_1 z_2) = \operatorname{Arg} z_1 + \operatorname{Arg} z_2$$

(up to multiples of 2π). Sketch on board. See also Example C.4.

In particular, if $z = r(\cos\theta + i\sin\theta)$, then $z^2 = r^2(\cos(2\theta) + i\sin(2\theta))$. It follows that the two **square roots** of z are

$$\pm\sqrt{r}(\cos(\theta/2) + i(\sin\theta/2))$$

The remaining material is for your interest only

Repeating this argument gives:

Theorem (De Moivre's Theorem): If $z = r(\cos\theta + i\sin\theta)$ and n is a positive integer, then

$$z^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

When $r \neq 0$, this also holds for n negative. In particular,

$$\frac{1}{z} = \frac{1}{r}\,(\cos\theta - i\sin\theta).$$

Example C.5: Find $(1+i)^6$.

Solution: We saw that $1 + i = \sqrt{2}(\cos(\pi/4) + i\sin(\pi/4))$. So

$$\begin{aligned} (1+i)^6 &= (\sqrt{2})^6(\cos(6\pi/4) + i\sin(6\pi/4)) \\ &= 8(\cos(3\pi/2) + i\sin(3\pi/2)) \\ &= 8(0 + i(-1)) = -8i \end{aligned}$$

nth roots

De Moivre's Theorem also lets us compute nth roots:

Theorem: Let $z = r(\cos\theta + i\sin\theta)$ and let n be a positive integer. Then z has exactly n distinct nth roots, given by

$$r^{1/n}\left[\cos\left(\frac{\theta + 2k\pi}{n}\right) + i\sin\left(\frac{\theta + 2k\pi}{n}\right)\right]$$

for $k = 0, 1, \ldots, n-1$.

These are equally spaced points on the circle of radius $r^{1/n}$.

Example: The cube roots of -8: Since $-8 = 8(\cos(\pi) + i\sin(\pi))$, we have

$$(-8)^{1/3} = 8^{1/3}\left[\cos\left(\frac{\pi + 2k\pi}{3}\right) + i\sin\left(\frac{\pi + 2k\pi}{3}\right)\right]$$

for $k = 0, 1, 2$. We get

$$2(\cos(\pi/3) + i\sin(\pi/3)) = 2(1/2 + i\sqrt{3}/2) = 1 + \sqrt{3}i$$

$$2(\cos(3\pi/3) + i\sin(3\pi/3)) = 2(-1 + 0i) = -2$$

$$2(\cos(5\pi/3) + i\sin(5\pi/3)) = 2(1/2 - i\sqrt{3}/2) = 1 - \sqrt{3}i$$

Euler's formula

Using some Calculus, one can prove:

Theorem (Euler's formula): For any real number x,

$$e^{ix} = \cos x + i\sin x$$

Thus e^{ix} is a complex number on the unit circle. This is most often used as a shorthand:

$$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

It also leads to one of the most remarkable formulas in mathematics, which combines 5 of the most important numbers:

$$e^{i\pi} + 1 = 0$$