Math 1600B Lecture 3, Section 2, 10 Jan 2014

Announcements:

Continue **reading** Section 1.2 for next class, as well as the code vectors part of Section 1.4. (The rest of 1.4 will not be covered.)

Work through recommended homework questions.

Tutorials start January 15, and include a **quiz** covering until Monday's lecture. More details on Monday.

Office hour: Monday, 1:30-2:30, MC103B.

Help Centers: Monday-Friday 2:30-6:30 in MC 106, but not starting until Monday, January 20.

Lecture notes (this page) available from course web page by clicking on the link. Answers to lots of administrative questions are available on the course web page as well.

Review of last lecture:

Many properties that hold for real numbers also hold for vectors in $\mathbb{R}^n$: Theorem 1.1. But we'll see differences later.

Definition: A vector $\vec{v}$ is a **linear combination** of vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ if there exist scalars $c_1, c_2, \ldots, c_k$ (called coefficients) such that

$$\vec{v} = c_1\vec{v}_1 + \cdots + c_k\vec{v}_k.$$

We also call the coefficients **coordinates** when we are thinking of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k$ as defining a new coordinate system.

Vectors modulo m:

$\mathbb{Z}_m = \{0, 1, \ldots, m-1\}$ with addition and multiplication taken modulo m. That means that the answer is the remainder after division by m.

For example, in $\mathbb{Z}_{10}$, $\quad 8 \cdot 8 = 64 = 4 \pmod{10}$.

$\mathbb{Z}_m^n$ is the set of vectors with n components each of which is in $\mathbb{Z}_m$.

To find solutions to an equation such as

$$6x = 6 \pmod{8}$$

you can simply try all possible values of x. In this case, 1 and 5 both work, and no other value works.

Note that you can not in general **divide** in $\mathbb{Z}_m$, only add, subtract and multiply.

Most of this course will concern vectors with real components. Vectors in $\mathbb{Z}_m^n$ will just be used to study code vectors.

New material

Section 1.2: Length and Angle: The Dot Product

Definition: The **dot product** of vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ is the real number defined by

$$\vec{u} \cdot \vec{v} := u_1 v_1 + \cdots + u_n v_n.$$

Since $\vec{u} \cdot \vec{v}$ is a *scalar*, the dot product is sometimes called the **scalar product**, not to be confused with *scalar multiplication* $c\vec{v}$.

The dot product will be used to define length, distance and angles in $\mathbb{R}^n$.

Example: For $\vec{u} = [1, 0, 3]$ and $\vec{v} = [2, 5, -1]$, we have

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 0 \cdot 5 + 3 \cdot (-1) = 2 + 0 - 3 = -1.$$

We can also take the dot product of vectors in $\mathbb{Z}_m^n$, by reducing the answer modulo m.

Example: For $\vec{u} = [1, 2, 3]$ and $\vec{v} = [2, 3, 4]$ in $\mathbb{Z}_5^3$, we have

$$\vec{u} \cdot \vec{v} = 1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 = 2 + 6 + 12 = 20 = 0 \pmod{5}.$$

In $\mathbb{Z}_6^3$, the answer would be $?$

Theorem 1.2: For vectors $\vec{u}, \vec{v}, \vec{w}$ in $\mathbb{R}^n$ and c in $\mathbb{R}$:
(a) $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
(b) $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$
(c) $(c\vec{u}) \cdot \vec{v} = c(\vec{u} \cdot \vec{v}) = \vec{u} \cdot (c\vec{v})$

(d) $\vec{u} \cdot \vec{u} \geq 0$
(e) $\vec{u} \cdot \vec{u} = 0$ if and only if $\vec{u} = \vec{0}$

Again, very similar to how multiplication and addition of numbers works.

Explain (b) and (d) on board. (a) and (c) are explained in text.

Length from dot product

The length of a vector $\vec{v} = [v_1, v_2]$ in $\mathbb{R}^2$ is $\sqrt{v_1^2 + v_2^2}$, using the Pythagorean theorem. (Sketch.) Notice that this is equal to $\sqrt{\vec{v} \cdot \vec{v}}$. This motivates the following definition:

Definition: The **length** or **norm** of a vector $\vec{v}$ in $\mathbb{R}^n$ is the scalar $\|\vec{v}\|$ defined by

$$\|\vec{v}\| := \sqrt{\vec{v} \cdot \vec{v}} = \sqrt{v_1^2 + \cdots + v_n^2}.$$

Example: The length of $[1, 2, 3, 4]$ is $\sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{30}$.

Note: $\|c\vec{v}\| = |c|\|\vec{v}\|$. (Explain on board.)

Definition: A vector of length 1 is called a **unit** vector.

The unit vectors in $\mathbb{R}^2$ form a circle. (Sketch.) Examples are $[1, 0]$, $[0, 1]$, $[\frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}]$, and lots more. The first two are denoted $\vec{e}_1$ and $\vec{e}_2$ and are called the **standard unit vectors** in $\mathbb{R}^2$.

The unit vectors in $\mathbb{R}^3$ form a sphere. The **standard unit vectors** in $\mathbb{R}^3$ are $\vec{e}_1 = [1, 0, 0]$, $\vec{e}_2 = [0, 1, 0]$ and $\vec{e}_3 = [0, 0, 1]$.

More generally, the **standard unit vectors** in $\mathbb{R}^n$ are $\vec{e}_1, \ldots, \vec{e}_n$, where $\vec{e}_i$ has a 1 as its ith component and a 0 for all other components.

Given any vector $\vec{v}$, there is a unit vector in the same direction as $\vec{v}$, namely

$$\frac{1}{\|\vec{v}\|} \vec{v}$$

This has length 1 using the previous Note. (Sketch and example on board.) This is called **normalizing** a vector.

Theorem 1.5: The Triangle Inequality: For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$\|\vec{u}+\vec{v}\| \le \|\vec{u}\| + \|\vec{v}\|.$$

On board: Example in $\mathbb{R}^2$: $\vec{u} = [1, 0]$ and $\vec{v} = [3, 4]$.

Theorem 1.5 is geometrically plausible, at least in $\mathbb{R}^2$ and $\mathbb{R}^3$. The book proves that it is true in $\mathbb{R}^n$ using Theorem 1.4, which we will discuss below.

Distance from length

Thinking of vectors $\vec{u}$ and $\vec{v}$ as starting from the origin, we define the **distance** between them by the formula

$$d(\vec{u}, \vec{v}) := \|\vec{u}-\vec{v}\| = \sqrt{(u_1 - v_1)^2 + \cdots + (u_n - v_n)^2},$$

generalizing the formula for the distance between points in the plane.

Example: The distance between $\vec{u} = [10, 10, 10, 10]$ and $\vec{v} = [11, 11, 11, 11]$ is

$$\sqrt{(-1)^2 + (-1)^2 + (-1)^2 + (-1)^2} = \sqrt{4} = 2.$$

Angles from dot product

The unit vector in $\mathbb{R}^2$ at angle θ from the x-axis is $\vec{u} = [\cos\theta, \sin\theta]$. Notice that

$$\vec{u} \cdot \vec{e}_1 = [\cos\theta, \sin\theta] \cdot [1, 0] = 1 \cdot \cos\theta + 0 \cdot \sin\theta = \cos\theta.$$

More generally, given vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^2$, one can show using the law of cosines that

$$\vec{u} \cdot \vec{v} = \cos\theta \, \|\vec{u}\| \, \|\vec{v}\|,$$

where θ is the angle between them (when drawn starting at the same point).

In particular, $|\vec{u} \cdot \vec{v}| \le \|\vec{u}\| \, \|\vec{v}\|$, since $|\cos\theta| \le 1$.

This holds in $\mathbb{R}^n$ as well, but we won't give the proof:

Theorem 1.4: The Cauchy-Schwarz Inequality: For all $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$,

$$|\vec{u} \cdot \vec{v}| \le \|\vec{u}\| \, \|\vec{v}\|.$$

We can therefore use the dot product to *define* the **angle** between two vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^n$ by the formula

$$\cos\theta := \frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}, \quad \text{i.e.,} \quad \theta := \arccos\left(\frac{\vec{u}\cdot\vec{v}}{\|\vec{u}\|\,\|\vec{v}\|}\right),$$

where we choose $0 \leq \theta \leq 180^\circ$. This makes sense because the fraction is between -1 and 1.

To help remember the formula for $\cos\theta$, note that the denominator normalizes the two vectors to be unit vectors.

On board: Angle between $\vec{u} = [1,2,1,1,1]$ and $\vec{v} = [0,3,0,0,0]$.

An applet illustrating the dot product. If it doesn't work, try the java version.

For a random example, you'll need a calculator, but for hand calculations you can remember these cosines:

$$\cos 0^\circ = \frac{\sqrt{4}}{2} = 1, \qquad \cos 30^\circ = \frac{\sqrt{3}}{2}, \qquad \cos 45^\circ = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}},$$

$$\cos 60^\circ = \frac{\sqrt{1}}{2} = \frac{1}{2}, \qquad \cos 90^\circ = \frac{\sqrt{0}}{2} = 0,$$

using the usual triangles.