Math 1600B Lecture 5, Section 2, 15 Jan 2014

Announcements:

Continue reading Section 1.3 and also the Exploration on cross products for next class. Work through recommended homework questions.

Quiz 1 this week in tutorials. See last lecture for how they run.

My office hour cancelled today.

Lecture notes (this page) available from course web page.

New material

[These notes are a summary of the material, which will be supplemented by lots of diagrams on the board.]

Section 1.3: Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

We study lines and planes because they come up directly in applications, but also because the solutions to many other types of problems can be expressed using the language of lines and planes.

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

Given a line ℓ, we want to find equations that tell us whether a point (x, y) or (x, y, z) is on the line. We'll write $\vec{x} = [x, y]$ or $\vec{x} = [x, y, z]$ for the position vector of the point, so we can use vector notation.

The **vector form** of the equation for ℓ is:

$$\vec{x} = \vec{p} + t\vec{d}$$

where $\vec{p}$ is the position vector of a point on the line, $\vec{d}$ is a vector parallel to the line, and $t \in \mathbb{R}$. This is concise and works in $\mathbb{R}^2$ and $\mathbb{R}^3$.

If we expand the vector form into components, we get the **parametric form** of the equations for ℓ:

$$\begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2 \\ (z &= p_3 + td_3 \quad \text{if we are in } \mathbb{R}^3) \end{aligned}$$

Lines in $\mathbb{R}^2$

There are additional ways to describe a line in $\mathbb{R}^2$.

The **normal form** of the equation for ℓ is:

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p},$$

where $\vec{n}$ is a vector that is *normal = perpendicular* to ℓ.

If we write this out in components, with $\vec{n} = [a, b]$, we get the **general form** of the equation for ℓ:

$$ax + by = c,$$

where $c = \vec{n} \cdot \vec{p}$. When $b \neq 0$, this can be rewritten as $y = mx + k$, where $m = -a/b$ and $k = c/b$.

Note: All of these simplify when the line goes through the origin, as then you can take $\vec{p} = \vec{0}$.

Example: Find all four forms of the equations for the line in $\mathbb{R}^2$ going through $A = [1, 1]$ and $B = [3, 2]$.

Note: None of these equations is *unique*, as $\vec{p}$, $\vec{d}$ and $\vec{n}$ can all change. The general form is closest to being unique: it is unique up to an overall scale factor.

Lines in $\mathbb{R}^3$

Most of the time, one uses the vector and parametric forms above. But there is also a version of the normal and general forms. To specify the direction of a line in $\mathbb{R}^3$, it is necessary to specify **two** non-parallel normal vectors $\vec{n}_1$ and $\vec{n}_2$. Then the **normal form** is

$$\vec{n}_1 \cdot \vec{x} = \vec{n}_1 \cdot \vec{p} \qquad \text{typo in book in Table 1.3:}$$
$$\vec{n}_2 \cdot \vec{x} = \vec{n}_2 \cdot \vec{p} \qquad \text{there should be no subscripts on } \vec{p}$$

When expanded into components, this gives the **general form**:

$$a_1 x + b_1 y + c_1 z = d_1,$$
$$a_2 x + b_2 y + c_2 z = d_2.$$

Since *both* equations must be satisfied, this can be interpreted as the intersection of two planes. (We'll discuss planes in a second.)

Question: What are the pros and cons of the different ways of describing a line?

Planes in $\mathbb{R}^3$

Normal form:

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}.$$

This is *exactly* like the normal form for the equation for a line in $\mathbb{R}^2$.
When expanded into components, it gives the **general form**:

$$ax + by + cz = d,$$

where $\vec{n} = [a, b, c]$ and $d = \vec{n} \cdot \vec{p}$.

Note: You can read off $\vec{n}$ from the general form. Two planes are parallel if and only if their normal vectors are parallel.

A plane can also be described in **vector form**. You need to specify a point $\vec{p}$ in the plane as well as two vectors $\vec{u}$ and $\vec{v}$ which are parallel to the plane but not parallel to each other.

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

When expanded into components, this gives the **parametric equations** for a plane:

$$\begin{aligned} x &= p_1 + su_1 + tv_1 \\ y &= p_2 + su_2 + tv_2 \\ z &= p_3 + su_3 + tv_3. \end{aligned}$$

Table 1.3 in the text summarizes this nicely (except for the one typo mentioned above).

It may seem like there are lots of different forms, but really there are two: vector and normal, and these can be expanded into components to give the parametric and general forms.

Example: Find all four forms of the equations for the plane in $\mathbb{R}^3$ which goes through the point $P = (1, 2, 0)$ and has normal vector $\vec{n} = [2, 1, -1]$.

Solution: For $\vec{p} = [1, 2, 0]$ and $\vec{n} = [2, 1, -1]$, we have $\vec{n} \cdot \vec{p} = 4$. So the normal form is

$$\vec{n} \cdot \vec{x} = 4.$$

The general form is

$$2x + y - z = 4.$$

To get the vector form, we need two vectors parallel to the plane, so we need two vectors perpendicular to $\vec{n}$. Can get these by trial and error, for example, $\vec{u} = [-1, 2, 0]$ and $\vec{v} = [0, 1, 1]$. Then the vector form is

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}.$$

Expanding into components gives the parametric form:

$$\begin{aligned} x &= 1 - s \\ y &= 2 + 2s + t \\ z &= \phantom{2 + 2s + {}} t. \end{aligned}$$

You can also find parallel vectors by finding two other points Q and R in the plane and then taking $\vec{u} = \overrightarrow{PQ}$ and $\vec{v} = \overrightarrow{PR}$. If $\vec{u}$ and $\vec{v}$ are parallel, you need to try again.

True/false: The planes given by

$$2x + 3y + 4z = 7$$

and

$$4x + 6y + 8z = 9$$

are parallel.

True, because the normal vectors are $[2, 3, 4]$ and $[4, 6, 8]$, which are parallel.

If the 9 was changed to 14, the two planes would be **equal**, but the answer would still be **true**, as a plane is parallel to itself. The right hand side shifts the position of a plane, but not its orientation.

True/false: The lines given by

$$\begin{aligned} \vec{x} &= \vec{p}_1 + t\vec{d}_1, \qquad \vec{p}_1 = [1, 2, 3], \qquad \vec{d}_1 = [2, 0, -2] \\ \vec{x} &= \vec{p}_2 + t\vec{d}_2, \qquad \vec{p}_2 = [2, 4, 6], \qquad \vec{d}_2 = [2, 1, 0] \end{aligned}$$

are parallel.

False, because the direction vectors are $[2, 0, -2]$ and $[2, 1, 0]$, which are not parallel. ($\vec{p}$ does not matter for this.)