Math 1600B Lecture 6, Section 2, 17 Jan 2014

Announcements:

Read Sections 2.0 and 2.1 for next class. Work through recommended homework questions.

The next quiz will be next week, and will cover the material until the end of Monday's class

Help Centers Monday-Friday 2:30-6:30 in MC 106 start on Monday, January 20.

Lecture notes (this page) available from course web page.

Partial review of last lecture:

Section 1.3: Lines and planes in $\mathbb{R}^2$ and $\mathbb{R}^3$

Lines in $\mathbb{R}^2$ and $\mathbb{R}^3$

The **vector form** of the equation for a line ℓ is:

$$\vec{x} = \vec{p} + t\vec{d},$$

where $\vec{p}$ is the position vector of a chosen point on the line, $\vec{d}$ is a vector parallel to the line, and $t \in \mathbb{R}$.

If we expand the vector form into components, we get the **parametric form** of the equations for ℓ:

$$\begin{aligned} x &= p_1 + td_1 \\ y &= p_2 + td_2 \\ (z &= p_3 + td_3 \quad \text{if we are in } \mathbb{R}^3) \end{aligned}$$

Lines in $\mathbb{R}^2$

For a line in $\mathbb{R}^2$, there are additional ways to describe a line.

The **normal form** of the equation for ℓ is:

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p},$$

where $\vec{n}$ is a vector that is *normal = perpendicular* to ℓ.

If we write this out in components, with $\vec{n} = [a, b]$, we get the **general form** of the equation for ℓ:

$$ax + by = c,$$

where $c = \vec{n} \cdot \vec{p}$. When $b \neq 0$, this can be rewritten as $y = mx + k$, where $m = -a/b$ and $k = c/b$.

Note: All of these simplify when the line goes through the origin, as then you can take $\vec{p} = \vec{0}$.

Note: None of these equations is *unique*, as $\vec{p}$, $\vec{d}$ and $\vec{n}$ can all change. The general form is closest to being unique: it is unique up to an overall scale factor.

Lines in $\mathbb{R}^3$

There are also **normal** and **general forms** of equations for a line in $\mathbb{R}^3$, which I won't review here.

Planes in $\mathbb{R}^3$

Normal form:

$$\vec{n} \cdot (\vec{x} - \vec{p}) = 0 \quad \text{or} \quad \vec{n} \cdot \vec{x} = \vec{n} \cdot \vec{p}.$$

When expanded into components, it gives the **general form**:

$$ax + by + cz = d,$$

where $\vec{n} = [a, b, c]$ and $d = \vec{n} \cdot \vec{p}$.

Vector form: You need to specify a point $\vec{p}$ in the plane as well as *two* vectors $\vec{u}$ and $\vec{v}$ which are parallel to the plane but not parallel to each other.

$$\vec{x} = \vec{p} + s\vec{u} + t\vec{v}$$

When expanded into components, this gives the **parametric equations** for a plane:

$$\begin{aligned} x &= p_1 + su_1 + tv_1 \\ y &= p_2 + su_2 + tv_2 \\ z &= p_3 + su_3 + tv_3. \end{aligned}$$

Table 1.3 in the text summarizes this material nicely.

It may seem like there are lots of different forms, but really there are two: vector and normal, and these can be expanded into components to give the parametric and general forms.

New material

Example: Find all four forms of the equations for the plane in $\mathbb{R}^3$ which goes through the points $P = (1, 1, 0)$, $Q = (0, 1, 2)$ and $R = (-1, 2, 1)$. Solution on board.

But we get stuck on the normal form, which motivates:

Cross products (Exploration after Section 1.3)

Given vectors $\vec{u}$ and $\vec{v}$ in $\mathbb{R}^3$, we would like a way to produce a new vector that is orthogonal to both $\vec{u}$ and $\vec{v}$. The **cross product** does this.

Definition: The **cross product** of $\vec{u}$ and $\vec{v}$ is the vector

$$\vec{u} \times \vec{v} := [u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3,\ u_1v_2 - u_2v_1].$$

Theorem: $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$. That is, $\vec{u} \cdot (\vec{u} \times \vec{v}) = 0$ and $\vec{v} \cdot (\vec{u} \times \vec{v}) = 0$.

Explain on board.

Note: The cross product only makes sense in $\mathbb{R}^3$!

Can now finish the previous example, on board.

Theorem: The cross product also has the following properties:
(a) $\vec{v} \times \vec{u} = -(\vec{u} \times \vec{v})$!
(b) $\vec{u} \times \vec{0} = \vec{0}$
(c) $\vec{u} \times \vec{u} = \vec{0}$
(d) $\vec{u} \times k\vec{v} = k(\vec{u} \times \vec{v}) = (k\vec{u}) \times \vec{v}$
(e) $\vec{u} \times k\vec{u} = \vec{0}$
(f) $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

This is exercise 5 from the cross product exploration, and I encourage you to check these properties.

Distance from a point to a line (back to Section 1.3)

Recall the formula for **the projection of $\vec{v}$ onto $\vec{u}$**:

$$\text{proj}_{\vec{u}}(\vec{v}) = \left(\frac{\vec{u} \cdot \vec{v}}{\vec{u} \cdot \vec{u}}\right)\vec{u}.$$

Example: Find the distance from the point $B = (1, 3, 6)$ to the line through $P = (1, 1, 0)$ in the direction $\vec{d} = [0, -1, 1]$. Solution on board, leading to $d(B, \ell) = \|\vec{v} - \text{proj}_{\vec{d}}(\vec{v})\| = 4\sqrt{2}$, where $\vec{v} = \overrightarrow{PB}$.

If the line was in $\mathbb{R}^2$ and had been described in normal form, one could instead compute $\|\text{proj}_{\vec{n}}(\vec{v})\|$, which saves one step.

Distance from a point to a plane

Example: Find the distance from the point $B = (1, 3, 6)$ to the plane $\mathcal{P}$ whose equation is $2x + y - z = 2$. Solution on board, leading to

$$d(B, \mathcal{P}) = \|\text{proj}_{\vec{n}}(\vec{v})\| = \frac{|\vec{n} \cdot \vec{v}|}{\|\vec{n}\|} = |-3|/\sqrt{6} = 3/\sqrt{6}.$$