Topology Vol. 32, No. 2, pp. 371-394, 1993.
Printed in Great Britain

0040-9383/93 $6.00 + 0.00


# UNIVERSAL PHANTOM MAPS

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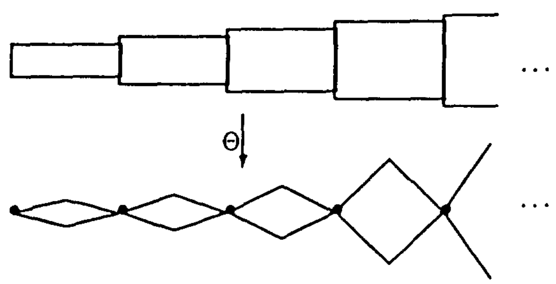
(Received 23 January 1992; in revised form 27 April 1992)

## §1. INTRODUCTION

LET $X$ be a connected $CW$-complex, with only a finite number of cells in each dimension, and let $X_n$ denote its $n$-skeleton. A map $f: X \to Y$ is called a *phantom* map if its restriction to each $X_n$ is null-homotopic. Non-trivial phantom maps are often elusive creatures that are hard to detect and difficult to visualize. There is one exception, however, and that is the subject of this paper. The universal phantom map out of $X$ is a map $\Theta: X \to \bigvee^{\infty} \Sigma X_n$ that can be viewed as follows. Identify the space $X$ with the direct limit of its $CW$-skeletons via the infinite telescope construction. Thus $X \simeq Tel(X)$ where

$$Tel(X) = \bigcup_{n \geq 1} X_n \times [n-1, n]/\sim$$

Here each $X_n \times \{n\}$ is identified with its image in $X_{n+1} \times \{n\}$. Now lay the telescope on its side and collapse to a point, each joint at which the second coordinate is an integer.



Then collapse to a point the seam along the basepoint in the target. The resulting map is a surjection from $Tel(X)$ to the infinite wedge of reduced suspensions

$$\bigvee^{\infty} \Sigma X_n = \Sigma X_1 \vee \Sigma X_2 \vee \Sigma X_3 \vee \ldots$$

It is easy to see that the map just described is a phantom map. Indeed, restrict it to the first $n$ stages of the telescope, and then deform that portion to the right into $X_n \times \{n\}$. This is a deformation retraction. Since $X_n \times \{n\}$ is sent to the base point in $\bigvee^{\infty} \Sigma X_n$, the assertion follows. Thus $\Theta$ is one phantom map which is easy to describe. We will show that the question of whether or not it is essential can, in many cases, be answered.

## §2. DESCRIPTION OF RESULTS

The main results of this paper are described and discussed in the next three sections. The proofs are deferred to the last section. Most of the results here deal with phantom maps out of those pointed spaces that are homotopy equivalent to $CW$-complexes of finite type. Such complexes have only a finite number of cells in each dimension. Of course, a space $X$ could have many different $CW$-decompositions and so the universal phantom map out of $X$, as we've defined it, is not unique. It depends on which finite type decomposition is chosen. We will assume, in what follows, that a choice has been made.

THEOREM 1. *If $X$ has finite type, then the map $\Theta$ is universal among phantom maps out of $X$. In other words, given another phantom map $f\colon X \to Y$, there exists a map $\bar{f}$ that makes the following diagram commute.*

$$\begin{array}{ccc} X & \xrightarrow{\ f\ } & Y \\ {\scriptstyle\Theta}\downarrow & \nearrow_{\bar{f}} & \\ \bigvee^{\infty} \Sigma X_n & & \end{array}$$

□

It follows, of course, that every phantom map out of $X$ is null homotopic if and only if $\Theta$ is. With regard to the dependence of $\Theta$ on the choice of $CW$-decomposition of $X$, this result suggests that one choice is as good as the next. The universal property in Theorem 1 leads to a very simple proof of the following.

COROLLARY 1.1. *If $f\colon X \to Y$ and $g\colon Y \to Z$ are two phantom maps where $X$ and $Y$ have finite type, then the composition $gf\colon X \to Z$ is null homotopic.* □

In recent years, many new examples of phantom maps have been discovered, thanks in large part, to the work of Meier [17], [18] and Zabrodsky [26]. The growing list of examples should cause one to suspect the presence of phantoms whenever dealing with a domain $X$, that has infinite dimensional homology and a range $Y$, that has infinitely many nonzero homotopy groups. It is clear that these conditions are necessary—at least among spaces of finite type, but they are not sufficient for the existence of essential phantoms. Indeed, there are some infinite dimensional spaces $X$, out of which all phantoms vanish! It is these spaces we study next.

THEOREM 2. *If $X$ has finite type, then all phantom maps out of $X$ are trivial if and only if $\Sigma X$ is a retract of $\bigvee^{\infty} \Sigma X_n$.* □

COROLLARY 2.1. *If $\Sigma X$ is equivalent to a bouquet of finite complexes, then the universal phantom map out of $X$ is trivial.* □

Since $\Sigma X$ is rationally equivalent to a bouquet of spheres, Corollary 2.1 implies the following result. In it, we follow Roitberg, [21], in using $Ph(X, Y)$, to denote the set of all homotopy classes of phantom maps from $X$ to $Y$.

COROLLARY 2.2. *If $X$ has finite type, then $Ph(X, Y) = 0$ for all rational spaces $Y$.* □

When $Y$ has finite type over $\mathbf{Q}$, this last result is well known and easy to prove using a $\varprojlim^1$ argument. The point is, of course, that this restriction on $Y$ is unnecessary. Here is a familiar loop space to which Corollary 2.1 applies.

EXAMPLE 2.3. *If $X$ has the homotopy type of $\Omega(\Sigma L_1 \times \ldots \times \Sigma L_n)$, where each $L_i$ is a connected finite complex, then $\Sigma X \simeq \bigvee_\alpha K_\alpha$ where each $K_\alpha$ is a finite complex. Thus all phantom maps out of such an $X$ are trivial.* □

## §3. WHEN IS $\Sigma X$ A RETRACT OF $\overset{\infty}{\bigvee} \Sigma X_n$?

It would be interesting to know the extent to which Corollary 2.1 characterizes those spaces $X$, for which the universal phantom map, $\Theta_X$, is null homotopic. This is the problem considered in this section. We begin by asking,

QUESTION 3. *If $\Sigma X$ is a retract of $\overset{\infty}{\bigvee} \Sigma X_n$, does it follow that $\Sigma X \simeq V_\alpha K_\alpha$ where each summand $K_\alpha$ is finite dimensional?*

The general answer to this question is no as the following example shows.

EXAMPLE 3.1. *There exists a CW-complex $X$, with the property that $\Sigma X$ is a retract of $\overset{\infty}{\bigvee} \Sigma X_n$ but $\Sigma X$ has no nontrivial finite dimensional retracts. Moreover, this space $X$ can be taken to have no odd dimensional cells, and at most one cell in each even dimension.* □

However, there are some special cases worth noting where the answer to Question 3 is yes!

PROPOSITION 3.2. *If $H_n(X; \mathbf{Z})$ is finite for each $n$ sufficiently large then the answer to Question 3 is yes. In other words, for such spaces $X$, the universal phantom map out of $X$ is trivial if and only if $\Sigma X$ decomposes into a bouquet of finite complexes.* □

Recall that an $H_o$-space is one that, when rationalized, becomes an $H$-space. Odd dimensional spheres, connected compact Lie groups, and complex Stiefel manifolds provide familiar examples of $H_o$-spaces. Notice that if $K$ is a 1-connected finite $CW$ complex and is also an $H_o$-space, then by Hopf's theorem it has the rational homotopy type of either a point or a finite product of odd dimensional spheres. The same is true of its double loop space, $\Omega^2 K$. In particular, this means that $\Omega^2 K$, which is almost always infinite dimensional, satisfies the hypothesis of Proposition 3.2. However, there are very few spaces $K$, that come to mind for which $\Omega^2 K$ splits apart into a bouquet of finite complexes after just one suspension. Consider, for example, the sphere $S^n$, with $n$ odd. While a theorem of Snaith asserts that $\Omega^2 S^n$ stably splits into an infinite bouquet of finite spectra, this splitting is not achieved after one suspension. In fact, there is no splitting of $\Sigma^t \Omega^2 S^n$ into a bouquet of finite complexes for any finite $t$, even when localized at the prime 2, according to Cohen and Mahowald [4]. Consequently, there must be essential phantom maps coming out of $\Omega^2 S^n$. We will say more about them later. But for now, let us localize at a prime $p$, and reconsider retracts of $\overset{\infty}{\bigvee} \Sigma X_n$. The next result suggests that Question 3 is a problem that is best studied one prime at a time.

THEOREM 3.3. *Let $X$ be a $CW$ complex with finite type and let $p$ be a fixed prime. Then $Ph(X, Y) = 0$ for all $p$-local spaces $Y$ if and only if $\Sigma X_{(p)}$ is equivalent to a bouquet of finite dimensional spaces.* □

Sometimes it is easy to rule out the existence of such splittings. Take, for example, the case where $H^*(X; \mathbb{Z}/p)$ is a polynomial algebra. The Steenrod algebra acts on this polynomial algebra in such a way that every nonzero orbit in positive degrees is an infinite set! As a result, $\Sigma X_{(p)}$ can have no finite dimensional retracts. In particular, when $X = BG$, the classifying space of a compact connected Lie group of rank $\geq 1$, it follows by taking $p$ sufficiently large, that there exist essential phantom maps out of $BG$.

It seems harder to verify the splitting of $\Sigma X_{(p)}$, when it occurs, than to rule it out when it doesn't. A case in point is the following conjecture, which seems beyond the reach of current techniques.

CONJECTURE 3.4. *If $K$ is a 1-connected finite complex, then for all primes sufficiently large, $\Sigma\Omega K_{(p)} \simeq \bigvee_\alpha F_\alpha$ where each $F_\alpha$ is finite dimensional.* □

Recall that a space $X$ is said to be atomic at a prime $p$, if any self map $f$, of its completion $\widehat{X_p}$, is either an equivalence or, under iteration, $f^n \to 0$ in the profinite topology on $[\widehat{X_p}, \widehat{X_p}]$. In particular, atomic spaces have no nontrivial idempotents and hence, they have no proper retracts.

COROLLARY 3.5. *Assume that $X$ has finite type. If for some prime $p$, either $\Sigma X_{(p)}$ or $\widehat{\Sigma X_p}$ has an infinite dimensional atomic retract, then the universal phantom map out of $X$ is nontrivial.* □

Here are some applications of this corollary.

EXAMPLE 3.6. *Let $G$ be a compact Lie group. Then the universal phantom map out of $BG$ is trivial if and only if $G$ is the trivial group.* □

The main ingredient in the verification of this example is Quillen's theorem that relates mod $p$ cohomology of $BG$ to the elementary abelian $p$-subgroups of $G$. The first nontrivial case, $G = \mathbb{Z}/2$, of this example appears in Lannes [14], p. 112. While Example 3.6 describes a very general phenomenon, the next one seems to be quite rare.

EXAMPLE 3.7. *Let $X = \Omega G$, where $G = Sp(2)$, $Sp(3)$, $G_2$ or $F_4$. In each case $X$ is stably atomic at the prime 2 and thus the universal phantom map out of it is essential. Indeed, it is stably essential.* □

The first two of the four loop spaces just mentioned, were shown to be stably atomic by M. Hopkins in his Northwestern Ph.D. thesis, [10]. J. Hubbuck then proved all four cases a different way in [12]. Of course, it is not always possible to translate questions about phantom maps into ones about stable homotopy. Indeed, we noted earlier that the universal phantom map out of $\Omega^2 S^n$, is essential, despite stable appearances to the contrary. The next example is similar in this respect although its proof is quite different.

EXAMPLE 3.8. *Let $X = \Omega SU(3)$. Then $X$ is stably equivalent to a bouquet of finite spectra, but the universal phantom map out of $X$ is essential at $p = 2$.* □

We suspect that this result holds for $\Omega SU(n)$, for all $n \geq 3$, but only the case $n = 3$ will be proved here. We trust the reader sees why $SU(2)$ must be excluded. A proof of the stable splitting of $\Omega SU(n)$ was given by M. Crabb and S. Mitchell in [6].

One might be tempted to conclude that if the universal phantom vanishes at every prime, then it must in fact be trivial. The next example shows this is not the case.

EXAMPLE 3.9.

$$Let\ X = \text{cofiber}\left\{\alpha_1 : \bigvee_{p\geq 3} S^{2p} \longrightarrow S^3\right\}$$

*where for each prime p,* $\alpha_1|S^{2p} = \alpha_1(p)$. *There is a phantom map* $X \to S^4$, *that is stably essential and yet the universal phantom map out of X is trivial at each prime p.* □

## §4. TARGETS OF FINITE TYPE

Since the outbound universal phantom map takes values in a space not of finite type, one might suspect that it is almost too sensitive; that it detects things never seen in a world of finite type. We will show that this suspicion is justified. To this end, we ask

QUESTION 4. *For what spaces X, is* $Ph(X, Y) = 0$ *for every target Y of finite type?*

When we speak of a target $Y$, of finite type, we mean that each $\pi_n Y$ is finitely generated, but we do not require the target to have the homotopy type of a $CW$-complex. There is nothing to be gained in doing so. For targets that happen to be $CW$-complexes, this condition on the homotopy groups is more restrictive than the one we use for domains $X$, of finite type; finite skeletons only imply that each $H_n(X; \mathbb{Z})$ is finitely generated. Of course, for simply connected $CW$-complexes, the two conditions are equivalent. Getting back to Question 4, the following example is perhaps an obvious one to consider. It is the only result in this paper wherein we drop the finite type restriction on the domain, $X$.

EXAMPLE 4.1. *Let X be a CW-complex whose homology groups,* $H_n(X; \mathbb{Z})$, *are torsion groups for all* $n \geq 1$. *Then* $Ph(X, Y) = 0$ *for every finite type target Y.* □

An important special case of this example is the classifying space, $BG$, for a finite group, $G$. Recall from Example 3.6 that the universal phantom map out of such a space is essential whenever $G \neq \{1\}$. So this is an instance where the sensitive nature of $\Theta$ is quite apparent. The next example represents the other extreme – with no torsion in its homology.

EXAMPLE 4.2. *Fix an odd prime p, and take the cofiber of Toda's α-family on* $S^3$. *To be more precise,*

$$let\ X = \text{cofiber}\left\{\alpha : \bigvee_{t\geq 1} S^{2t(p-1)+2} \longrightarrow S^3\right\}$$

*where for each t,* $\alpha|S^{2t(p-1)+2} = \alpha_t$. *Then the universal phantom map out of X is essential, but again* $Ph(X, Y) = 0$ *for all targets Y, of finite type.* □

The next theorem exploits a feature common to both examples; namely the existence of a less complicated space $Z$, whose universal phantom is trivial, and map $X \to Z$ that induces an isomorphism in rational homology. In the first example $Z$ would be a point, of course. In the second, it would be a bouquet of spheres.

THEOREM 4.3. *Let* $f: X \to Z$ *be a map that induces a rational homology isomorphism between CW-spaces of finite type. If Y is a finite type target, then* $Ph(Z, Y) = 0 \Rightarrow Ph(X, Y) = 0$. □

Meier announced a result like this in [17], with the extra hypothesis that $Y$ be an $H_o$-space. As far as we know, no proof of it was published.

COROLLARY 4.4. *Let $f: X \to Z$ be as in 4.3. If the universal phantom map out of $Z$ is null homotopic, then $Ph(X, Y) = 0$ for all finite type targets $Y$.* □

The map $f$ plays a crucial role in this corollary; reverse its direction and the result becomes false. To see this, consider $S^3 \to K(\mathbf{Z}, 3)$ or $\Omega S^3 \to \mathbf{CP}^\infty$. In both cases the domain would be a suitable candidate for $Z$ and the maps could be taken to be rational equivalences. However, it is easy to check that there are no maps going in the opposite direction to play the role of $f$. Of course, this is as it should be since essential phantom maps are known to exist from a $K(\mathbf{Z}, n)$ to $S^{n+1}$, [8].

Given a 1-connected finite $CW$-complex $M$, it is a simple matter to construct an appropriate product of odd dimensional spheres and loop spaces of other odd dimensional spheres, along with a map,

$$\prod_\alpha S^{2n_\alpha+1} \times \prod_\beta \Omega S^{2n_\beta+1} \xrightarrow{g} \Omega M,$$

that induces an isomorphism in rational homology. Finding a map in the opposite direction that induces a rational homology isomorphism is, however, quite another matter. It is a central unsolved problem in unstable homotopy theory. The following corollary deals with one of the few classes of compact spaces for which one knows that such a map exists.

COROLLARY 4.5. *Let $K$ be a 1-connected finite $H_0$-space. Then $Ph(\Omega K, Y) = 0$ for any finite type space $Y$.* □

Let $Aut(X)$ denote the discrete group of homotopy classes of self equivalences of $X$ and let $X^{(n)}$ denote the Postnikov approximation of $X$ up through dimension $n$. There is a short exact sequence of groups,

$$WI(X) \longrightarrow Aut(X) \longrightarrow \lim_{\leftarrow} Aut\, X^{(n)},$$

in which the first group, $WI(X)$, consists of those self equivalences of $X$ that become homotopic to the identity when restricted to any finite skeleton of $X$. Roitberg has shown that $Ph(X, X)$ and $WI(X)$ are isomorphic as groups when $X$ is a homotopy associative $H$-space, [21]. These facts, together with Corollary 4.5, imply the next result.

COROLLARY 4.6. *Let $K$ be as in Corollary 4.5. Then $WI(\Omega K) = 0$.* □

Our last example combines Corollary 4.4 with some deep results of Cohen, Moore and Neisendorfer on the homotopy groups of spheres and the double suspension.

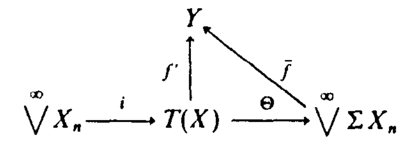
EXAMPLE 4.7. *There is an essential phantom map $\Omega^2 S^5 \to \mathbf{HP}^\infty$, and yet for every prime $p$ and every $n \geq 1$, $Ph(\Omega^2 S^{2n+1}, Y_{(p)}) = 0$ for every nilpotent target $Y$ of finite type.* □

We have been concerned, in this paper, with phantom maps going out of a given space $X$. There is an Eckmann–Hilton dual problem which considers phantom maps coming into a given space $Y$. In another paper we will show that while the universal *outbound* phantom map, $\Theta$, has an Eckmann–Hilton dual, some of its most important properties do not dualize.

Before giving the proofs, we would like to make a few remarks about the history of this topic. A. Heller was one of the first to consider phantom maps. It was he who named them! J. F. Adams and G. Walker gave the first published account of an essential phantom map—from $\Sigma\mathbf{CP}^\infty$ to an infinite bouquet of 4-spheres. It was in response to a question from P. Olum and appeared in [1]. One of the first detailed studies of phantom maps was done by B. Gray in his University of Chicago Ph.D thesis, [7], written under the direction of M. Barratt. Some of the results in this paper were first discovered there. The notion of a universal phantom map can be found there but it also appears in a paper of J. Lannes [14]. Letting $B$ denote $\mathbf{RP}^{\infty}$, Lannes mentions in passing that the universal phantom map out of $B$ is an example which shows that the restriction to finite type spaces in his theorem, $[B, Y] \approx Hom_{\kappa}(H^*Y, H^*B)$, is necessary. We thank Joe Neisendorfer for bringing Lannes's example to our attention and also for his help on some key points in this paper. Thanks go to Fred Cohen for suggesting the examples $\Omega^2 S^n$ and $\Omega SU(3)$, as well as the methods for dealing with them.

## §5. PROOFS

*Proof of Theorem* 1. Take the telescope $Tel(X)$, described at the outset, and collapse to a point its seam along the basepoint. Call the quotient space $T(X)$. Now identify the $n$-skeleton of $X$ with the image of $X_n \times \{n-1\}$ in $T(X)$. This defines an inclusion $i\colon \bigvee^{\infty} X_n \to T(X)$ which is easily seen to be a cofibration. There is also an obvious homotopy equivalence $\pi\colon T(X) \to X$, induced by projection on the first factor. Given a phantom map $f\colon X \to Y$, let $f' = f\pi$ in the following diagram




Since the restriction of $f'$ is null homotopic on each $X_n$, there is an extension $\bar{f}$, to the cofiber of $i$, and so the result follows. □

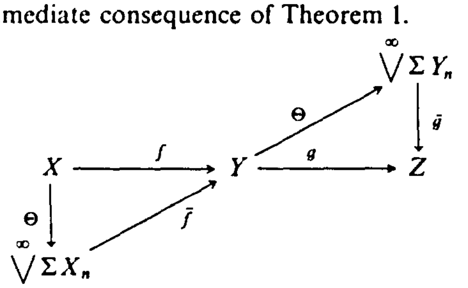
In the proofs that follow we will identify $T(X)$ with $X$ by means of the equivalence $\pi$. The corresponding map of $\bigvee^{\infty} X_n$ into $X$ will be denoted as the folding map, $\mathscr{F}$. Its restriction to each $X_n$ is the standard inclusion.

It might be worth considering another proof of Theorem 1 in terms of the identification

$$Ph(X, Y) \approx \varprojlim{}^1 [\Sigma X_k, Y]$$

The $\varprojlim{}^1$ term is a quotient of $\prod [\Sigma X_k, Y]$, and so for each phantom map $f\colon X \to Y$, one can choose a representative sequence $f_k\colon \Sigma X_k \to Y$. In particular, when $Y = \bigvee^{\infty} \Sigma X_n$ and $f = \Theta$, the representatives $f_k$ can be taken to be the standard inclusions $\Sigma X_k \to \bigvee^{\infty} \Sigma X_n$. Thus, in this setting the universal property of $\Theta$ is also apparent.

*Proof of Corollary* 1.1. The following commutative diagram, in which $f$ and $g$ are phantom maps, is an immediate consequence of Theorem 1.



The composition going up the diagonal is a phantom because the second map, $\Theta$, is. The restriction of this composition to each $\Sigma X_n$ is therefore null homotopic since each summand is a finite complex. Since a map out of a bouquet is completely determined by such restrictions, we conclude the map along the diagonal is trivial. This, of course, implies that the horizontal composition $gf$ must likewise be null homotopic. □

*Proof of Theorem* 2. We begin with the cofiber sequence

$$\bigvee^{\infty} X_n \xrightarrow{\mathscr{F}} X \xrightarrow{\Theta} \bigvee^{\infty} \Sigma X_n$$

in which $\mathscr{F}$ is the folding map, as before. Now pass three places to the right in the Barratt–Puppe sequence and consider

$$\xrightarrow{\Theta} \bigvee^{\infty} \Sigma X_n \xrightarrow{\delta} \bigvee^{\infty} \Sigma X_n \xrightarrow{\Sigma\mathscr{F}} \Sigma X \longrightarrow$$

In this sequence, the map $\Theta$ is null homotopic if and only if the next map $\delta$ has a left inverse. This, in turn holds if and only if the next map $\Sigma\mathscr{F}$ has a right inverse. Recall that with either inverse one can construct an equivalence

$$\bigvee^{\infty} \Sigma X_n \simeq \left(\bigvee^{\infty} \Sigma X_n\right) \vee \Sigma X$$

and conversely, from such an equivalence that is compatible with the maps $\delta$ and $\Sigma\mathscr{F}$, one can obtain the required inverses. Thus it is clear that if $\Theta$ is trivial, then $\Sigma X$ is a retract of $\bigvee^{\infty} \Sigma X_n$ and $\Sigma\mathscr{F}$ is a retraction. Going the other way, assume that we have maps

$$\Sigma X \xrightarrow{i} \bigvee^{\infty} \Sigma X_n \xrightarrow{r} \Sigma X$$

such that $ri$ is homotopic to the identity on $\Sigma X$. Let $r_n$ denote the restriction of $r$ to $\Sigma X_n$. By the cellular approximation theorem we can assume that each $r_n$ is homotopic to a self map $\rho_n$ of $\Sigma X_n$ followed by the standard inclusion of $\Sigma X_n$ in $\Sigma X$. Then

$$1_X \simeq ri \simeq \Sigma\mathscr{F}\left(\bigvee^{\infty} \rho_n\right) i$$

Thus $\Sigma\mathscr{F}$ has a right inverse and so $\Theta$ is trivial by earlier remarks. □

*Proof of Corollary* 2.1. Assume that $\Sigma X \simeq \bigvee_\alpha K_\alpha$, where each $K_\alpha$ is a finite complex. For each $\alpha$ choose an embedding $j_\alpha \colon K_\alpha \to \Sigma X_{n_\alpha}$ that has a left inverse. Choose them so that $n_\alpha \neq n_\beta$ when $\alpha \neq \beta$. There is then an obvious left inverse for the composition

$$\Sigma X \simeq V_\alpha K_\alpha \xrightarrow{V_\alpha j_\alpha} V_\alpha \Sigma X_{n_\alpha} \subseteq \bigvee^{\infty} \Sigma X_n$$

and so the result follows. □

*Proof of Corollary* 2.2. The space $X$ may not be nilpotent, but we can certainly rationalize everything to the right of it in the cofiber sequence considered earlier,

$$X \xrightarrow{\Theta_o} \left(\bigvee^{\infty} \Sigma X_n\right)_o \xrightarrow{\delta_o} \left(\bigvee^{\infty} \Sigma X_n\right)_o \longrightarrow (\Sigma X)_o \longrightarrow \dots$$

Since $\Sigma X$ is rationally a bouquet of spheres, the map $\delta_0$ has a left inverse, and so the first map $\Theta_0$ is trivial. Since any phantom map from $X$ to a rational space $Y$ factors through $\Theta_0$, the result follows. □

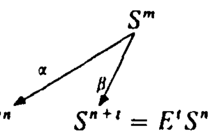
As mentioned earlier, this result can also be verified using a $\varprojlim^1$ argument, provided $Y$ has finite type over Q. In this case, the groups $[\Sigma X_k, Y]$ will be finite dimensional rational vector spaces, and the maps in the tower will be linear. Consequently, the descending chain condition on vector subspaces forces the tower to be Mittag-Leffler and so its $\varprojlim^1$ term must vanish.

*Proof of Example* 2.3. When there is just one factor, a classical result of I. M. James asserts that

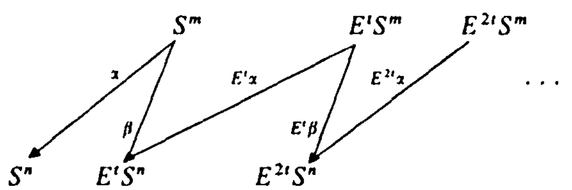
$$\Sigma\Omega\Sigma L \simeq \Sigma(L \vee L \wedge L \vee L \wedge L \wedge L \vee \cdots)$$

and so if $L$ is a finite complex, then so are all the other summands. If there is more than one factor, use the equivalences, $\Omega(X \times Y) \simeq \Omega X \times \Omega Y$, and $\Sigma(X \times Y) \simeq \Sigma X \vee \Sigma Y \vee \Sigma(X \wedge Y)$, and induction on the number of factors to prove it for the product. □

*Proof of Example* 3.1. The construction of this example begins with two homotopy classes of essential maps, say $\alpha$ and $\beta$,




whose orders are finite, stable, and relatively prime, and whose targets are different. In other words, if $\|\alpha\|$ denotes the order of $\alpha$, we require that $\|\alpha\| = \|E^n\alpha\|$ for all $n$, the same condition for $\beta$, and $(\|\alpha\|, \|\beta\|) = 1$. We also want $t > 0$. For instance, we could take $\alpha = \alpha_1(7)$ in $\pi_{14}S^3$ and $\beta = E^8\alpha_1(3)$ in $\pi_{14}S^{11}$ in this construction. We will construct a cell complex using the following suspensions of these maps




More precisely we take the space $X$ in this example to be the mapping cone,

$$X = \text{cofiber}\left\{\Psi\colon \bigvee_{k\geq 0} E^{kt}S^m \longrightarrow \bigvee_{k\geq 0} E^{kt}S^n\right\}$$

where the restriction of $\Psi$ to $E^{kt}S^m$ is given by

$$E^{kt}\{S^m \xrightarrow{\nu} S^m \vee S^m \xrightarrow{\alpha\vee\beta} S^n \vee S^{n+t}\} \xrightarrow{\subseteq} \bigvee_{j\geq 0} E^{jt}S^n$$

The first map here is the standard comultiplication on the sphere—the one that pinches the equator to a point.

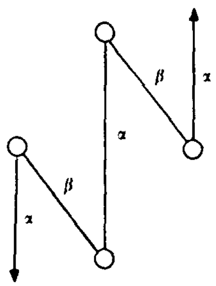
We have claimed that $\Sigma X$ has no proper finite dimensional retracts. This is easy to see in the example mentioned earlier in which $\alpha$ and $\beta$ are detected by primary Steenrod operations. We will show it is true more generally whenever $\alpha$ and $\beta$ are stably essential and their orders are relatively prime. To keep the notation simple, we will deal with the space $X$ and prove that it is irreducible. The method however will give the same result for any suspension of $X$.

By way of a contradiction, suppose that $K$ is a finite dimensional retract of $X$, and that $K$ has dimension $d$, which is positive but minimal. As $K$ is a retract, its homology is isomorphic to a direct summand of $H_*X$. Since each betti number of $X$ is either 0, 1 or possibly 2 (iff $m - n \equiv -1 \mod t$), the same is true for $K$. Let us start with the case where $H_dK$ has rank 1. Then the inclusion $K \to X$ would take the top cell of $K$ to either a sphere $E^{kt}S^n$, or to a cone on one of the $E^{kt}S^m$'s or possibly to a linear combination of the two. Now if the top cell of $K$ is sent to a spherical class, and $K$ is minimal, then this forces $K = E^{kt}S^n$. But when localized at the order of $\alpha$

$$X \simeq \bigvee_{k\geq 0} E^{kt}(S^n \cup_\alpha e^{m+1})$$

and to have one of the bottom spheres in these mapping cones as a retract would imply that $E^{kt}\alpha$ is nullhomotopic—a contradiction.

Next assume that the top cell of $K$ is sent to the cone over $E^{kt-t}S^m$, where $k \geq 1$. When localized at the order of $\beta$, this cone remains attached to the sphere $E^{kt}S^n$ by the stably essential map $E^{kt-t}\beta$. This means that the sphere $E^{kt}S^n$ is contained in $K$. But it also means that when localized at the order of $\alpha$, this same sphere is a retract of the mapping cone of $E^{kt}\alpha$—a contradiction once again.




Finally assume that the top class in $H_*K$ is sent to a linear combination of two classes in $H_*X$ and thus $d = m + jt + 1 = n + kt$. Choose generators, $x$ and $y$ in $H_dX$ representing the cone and the sphere respectively. If the top class of $K$ is sent to $\lambda x + \mu y$, then this element generates a direct summand of $H_dX$ and so the coefficients $\lambda$ and $\mu$ must be relatively prime. Suppose there is a prime $p$, that divides the stable order of $\alpha$, but does not divide $\mu$. If such a prime exists, localize $X$ at it. As before $X_{(p)}$ will decompose into an infinite wedge of mapping cones for the various suspensions of $\alpha$ and the retraction of $X$ to $K$, restricted to the appropriate mapping cone will yield a retraction of the bottom sphere—and a contradiction. The same argument can be used if $p$ divides the stable order of $\beta$ and again does not divide $\mu$. The only other possibility is that the coefficient $\lambda$ is relatively prime to the orders of $\alpha$ and $\beta$. If this happens we recall the argument used when the top class of $K$ was mapped to a nonspherical generator. It will work here too. There is one more case to be considered; when $H_*K$ has rank 2 in its top dimension. This case is easy and is left to the reader.

In the proof that $\Sigma X$ is a retract of $\bigvee^{\infty} \Sigma X_n$, we will abuse notation slightly and use $\alpha + \beta$ to refer to the following composition,

$$S^m \xrightarrow{\nu} S^m \vee S^m \xrightarrow{\alpha \vee \beta} S^n \vee S^{n+t}$$

Its mapping cone will be denoted $C(\alpha+\beta)$. We will construct a map

$$X \longrightarrow \bigvee_{k\geq 0} E^{kt}C(\alpha+\beta)$$

with a left inverse. From this it will follow easily that $\Sigma X$ is a retract of $\overset{\infty}{\bigvee}\Sigma X_n$. To begin the construction, first choose integers $a$ and $b$ such that

$$a\|\alpha\| + b\|\beta\| = 1.$$

This is possible since the orders of $\alpha$ and $\beta$ are relatively prime. Consider next the $t$-fold suspension of the composition,

$$S^{m-t} \vee S^m \xrightarrow{(E^{-t}\beta,\,\alpha)} S^n \xrightarrow{\nu} S^n \vee S^n \xrightarrow{a\|\alpha\| \vee b\|\beta\|} S^n \vee S^n.$$

Call this composition $\Phi$. When followed by the folding map $\mathscr{F}$ on $S^{n+t}$, one gets $\mathscr{F}\Phi = (\beta, E^t\alpha)$ because the last three maps in that composition factor the identity on $S^{n+t}$. When $\Phi$ is composed with the projection onto the left $S^{n+t}$, the result is $\pi_L\Phi = (\beta, 0)$, while projection onto the right sphere yields $\pi_R\Phi = (0, E^t\alpha)$. These facts are easy to verify when $\beta$ is a co-$H$ map. So, from this point on, let us assume that it is. If $\iota$ denotes the identity map on $S^{n+t}$, the following relations in $\pi_* S^{n+t}$ become evident.

$$(b\|\beta\|\iota)\beta = 0, \quad \text{and}$$

$$(a\|\alpha\|\iota)\beta = \beta$$

since $a\|\alpha\| \equiv 1 \bmod \|\beta\|$. These relations are at the heart of the second and third assertions about $\Phi$. Now consider the commutative diagram,

$$\begin{array}{ccccc}
\bigvee_{k\geq 0} E^{kt}S^m & \xrightarrow{\Psi} & \bigvee_{k\geq 0} E^{kt}S^n & \longrightarrow & X \\
\downarrow & & \downarrow & & \downarrow \\
\bigvee_{k\geq 0} E^{kt}S^m & \xrightarrow{\bigvee E^{kt}(\alpha+\beta)} & \bigvee_{k\geq 0} E^{kt}(S^n \vee S^{n+t}) & \longrightarrow & \bigvee_{k\geq 0} E^{kt}\, C(\alpha+\beta)
\end{array}$$

The rows in this diagram are cofiber sequences. The vertical map on the left is the identity. The map in the middle resticts to the identity on the bottom $S^n$, but for $k \geq 1$, it sends $E^{kt}S^n$ to a bouquet of two spheres of dimension $n + kt$, via the $kt$-fold suspension of the composition,

$$S^n \xrightarrow{(a\|\alpha\| \vee b\|\beta\|)\nu} S^n \vee S^n \xrightarrow{i_R \vee i_L} E^{-t}(S^n \vee S^{n+t}) \vee E^0(S^n \vee S^{n+t})$$

Therefore, a typical sphere $E^{kt}S^m$, in the upper left corner is sent to $E^{kt}(\alpha+\beta)$ along either route and so the left square commutes. The right-hand map is, of course, the quotient map on the cofibers. It is clear this map is injective in homology.

The mapping cone $E^{kt}C(\alpha+\beta)$ can be embedded in the $kt + m + 1$-skeleton of $X$, by a map that is injective in homology. The easiest way to see this is to return to the picture of the attaching maps for $X$, given earlier. To construct $X$, we start with the spheres, $\bigvee E^{kt}S^n$, and attach cells by the maps $E^{kt}(\alpha+\beta)$. These cells can be attached in any order we choose. Attaching the cell of dimension $kt + m + 1$ first, we get the embedding in question.

Now suspend once and compose the two embeddings just described.

$$\Sigma X \longrightarrow \bigvee_{k\geq 0} E^{kt+1}C(\alpha+\beta) \longrightarrow \overset{\infty}{\bigvee}\Sigma X_n$$

Follow this by the folding map, $\mathscr{F}: \bigvee^{\infty} \Sigma X_n \to \Sigma X$. It rejoins those spherical classes split apart by the $(a\|\alpha + b\|\beta\|)\nu$-map. The three maps compose to a self equivalence of $\Sigma X$, and so this space is a retract of $\bigvee^{\infty} \Sigma X_n$, as claimed. $\square$

*Proof of Proposition* 3.2. One direction of this result is covered by Corollary 2.1. To begin the proof in the other direction, assume that $\Sigma X$ is a retract of $\bigvee^{\infty} \Sigma X_n$. In the proof of Theorem 2, it was shown that the inclusion $j$, could be chosen to be a right homotopy inverse to the folding map, $\mathscr{F}$. Thus the composition

$$\Sigma X \xrightarrow{j} \bigvee^{\infty} \Sigma X_n \xrightarrow{\mathscr{F}} \Sigma X$$

is homotopic to the identity on $\Sigma X$.

Since the integral homology groups of $\Sigma X$ are finitely generated, one can find for any $q$, an integer $t$ such that there is a monomorphism along the diagonal that makes the following diagram commute

$$\begin{array}{ccc} H_q(\Sigma X) & \xrightarrow{j_*} & H_q(\bigvee^{\infty} \Sigma X_n) \\ & \searrow & \uparrow (i_t)_* \\ & & H_q(\bigvee_{n=1}^{t} \Sigma X_n) \end{array}$$

Let $\pi_t: \bigvee^{\infty} \Sigma X_n \to \bigvee_{n=1}^{t} \Sigma X_n$ be the projection—the left inverse to the inclusion $i_t$ in the diagram. Choose the integer $t$ large enough to ensure that the following composition induces the identity in homology through a range of dimensions, $d$, above which each $H_n \Sigma X$ is finite.

$$\Sigma X \xrightarrow{j} \bigvee^{\infty} \Sigma X_n \xrightarrow{\pi_t} \bigvee_{n=1}^{t} \Sigma X_n \xrightarrow{\mathscr{F}_t} \Sigma X$$

Let $\varphi_t$ denote this composition. In $H_*(\Sigma X, \mathbf{Z})$ this map induces the identity in degrees $\leq d$ and the zero map in degrees $\geq t$. In those degrees between $d$ and $t$, the homology groups of $\Sigma X$ are finite. Therefore some iterate of $\varphi_t$ induces an idempotent on $H_*(\Sigma X, \mathbf{Z})$. This follows from the observation that, given an endomorphism of a finite set, some iterate of it must be an idempotent. In the case at hand, let us simply refer to the appropriate iterate as $\varphi$. Take the telescope construction,

$$K = Tel\{\Sigma X \xrightarrow{\varphi} \Sigma X \xrightarrow{\varphi} \Sigma X \xrightarrow{\varphi} \cdots\},$$

whose homology realizes the image of $\varphi_*$. The inclusion of $\Sigma X$ in the left end of the telescope gives a map $\Sigma X \to K$ which is homology epimorphism. Use the suspension co-$H$-structure on $\Sigma X$, to form the self-map $1 - \varphi$, and let $L$ denote the telescope of this homology idempotent. Then add the two projections together to obtain a homology equivalence,

$$\Sigma X \longrightarrow K \vee L$$

and, hence a homotopy equivalence, as the three spaces are simply connected. Notice that the space $L$ is $d$-connected and that all of its reduced homology groups are finite. This implies that the argument just used to split a finite dimensional retract off $\Sigma X$ can be used again to do the same to $L$. The appropriate homology idempotent can be obtained as the composition

$$L \longrightarrow \Sigma X \longrightarrow \bigvee^{\infty} \Sigma X_n \longrightarrow \bigvee_{n=1}^{\tau} \Sigma X_n \longrightarrow \Sigma X \longrightarrow L$$

where $\tau > t$. Care should be taken here to choose the inclusion on the left and the retraction on the right to be compatible with the self map $1 - \varphi$. In addition, the integer $\tau$ should be taken large enough to ensure that, in this composition of five maps, the middle three induce the identity on $\Sigma X$ up through dimension $t$. Repeating this procedure over and over, and then taking limits, it follows that

$$\Sigma X \simeq \bigvee_{\alpha} K_{\alpha}$$

where each summand $K_{\alpha}$ is a finite complex. □

*Proof of Theorem* 3.3. In this proof, the space $\Sigma X$ will be assumed to be $p$-local, but the notation will not be burdened with this assumption. Reduced integral homology will be used throughout. The initial goal of the proof is to split $\Sigma X$ into two pieces, say

$$\Sigma X \simeq K \vee L$$

where $K$ is finite dimensional and the connectivity of $L$ is strictly greater than that of $\Sigma X$. To this end, we can assume, without any real loss of generality, that the first nonzero homology group of $\Sigma X$ occurs in degree 2. By hypothesis, there is an inclusion

$$j: \Sigma X \longrightarrow \bigvee^{\infty} \Sigma X_n$$

that is a right inverse to the folding map $\mathscr{F}: \bigvee^{\infty} \Sigma X_n \rightarrow \Sigma X$. Since $H_2 \Sigma X$ is a finitely generated $\mathbf{Z}_{(p)}$-module, its image under $j_*$ is contained in a finitely generated summand, say

$$H_2\Big(\bigvee_{1 \le n \le t} \Sigma X_n\Big) \subseteq H_2\Big(\bigvee^{\infty} \Sigma X_n\Big)$$

Let $\phi: \Sigma X \rightarrow \Sigma X$ be the following composition, just as in the proof of 3.2.

$$\Sigma X \xrightarrow{j} \bigvee^{\infty} \Sigma X_n \xrightarrow{\pi_t} \bigvee_{1 \le n \le t} \Sigma X_n \xrightarrow{\mathscr{F}_t} \Sigma X$$

Note that in homology $\phi$ induces the identity in degree 2 and the zero map in degrees greater than $t$. If $\phi$ induces a pseudoprojection—that is, a homomorphism $h$ such that $image(h) = image(h^2)$—in all remaining degrees as well, then we will use it to form the telescope

$$Tel\{\Sigma X \xrightarrow{\phi} \Sigma X \xrightarrow{\phi} \Sigma X \xrightarrow{\phi} \dots\}$$

whose homology realizes the image of $\phi_*$. This telescope will have finite dimensional homology. It will be our $K$. The telescope corresponding to $1 - \phi$ will be $L$. The initial splitting will have been achieved.

If $\phi$ does not induce a pseudoprojection in some degree $d$, where $2 < d \le t$, then there is work to be done. We will follow Wilkerson, [25], in obtaining the desired pseudoprojection. However, since his results pertain to finite dimensional spaces, a few changes are needed for our purposes.

Let $H = H_{\leq t}\Sigma X$, let $T$ denote the torsion subgroup of $H$, and let $V = H/T$. Since $V$ has finite rank, we can assume that if $r > 0$ is given, then some iterate of $\varphi$ will induce an idempotent on the finite set, $V \otimes \mathbf{Z}/p^r$. Wilkerson proves the following algebraic fact in Step 1 of his Theorem 3.3: if $B$ is an endomorphism of $V$ that induces an idempotent on $V \otimes \mathbf{Z}/p^r$, then there exists a pseudoprojection $B'$ on $V$, such that

$$B' = B + p^r A.$$

We will combine this fact with the following result—the analogue of Wilkerson's Theorem 3.2.

LEMMA 3.3.1. *There is an integer $r > 0$, such that if $A$ is any endomorphism of the graded $\mathbf{Z}_{(p)}$ module $V$, then $p^r A$ is realizable by a self-map of $\Sigma X$. Moreover this self-map can be taken to induce the zero map in degrees greater than $t$.* □

Assume for the moment that this lemma is true. Take $r$ large enough to satisfy the condition in the lemma, take $B = \phi_*/\text{torsion}$, and let $\alpha: \Sigma X \to \Sigma X$ be the map that realizes $p^r A$ on $V$. Use the suspension co-$H$-structure on $\Sigma X$ to form the sum,

$$\psi = \phi + \alpha: \Sigma X \longrightarrow \Sigma X$$

Then on $H_*\Sigma X$, the self-map $\psi$ induces the identity in degree 2, the zero map in degrees greater than $t$, and a pseudoprojection on $V$.

We claim that some iterate of $\psi$ induces a pseudoprojection on $H$, and hence on all of $H_*\Sigma X$. To simplify the notation in the proof of this claim, let $f$ denote the endomorphism of $H$ induced by $\psi$. Since the torsion subgroup $T$ is finite, it is clear that the nested sequence of subgroups $\{T \cap \mathit{image}(f^n)\}$ eventually stabilizes. Thus for $n$ sufficiently large,

$$T \cap \mathit{image}(f^n) = T \cap \mathit{image}(f^{2n})$$

Now if $f$ induces a pseudoprojection on $V$, then so does $f^n$. This means that for any element $y \in H$,

$$f^n(y) = f^{2n}(x) + z$$

for some $x \in H$, and $z \in T$. This equation implies that $z$ is in the image of $f^n$, and hence in $\mathit{image}(f^{2n})$, when $n$ is sufficiently large. In this case, the claim

$$\mathit{image}(f^n) = \mathit{image}(f^{2n}), \quad \text{on } H$$

follows.

We have shown that $\psi^n$, for $n$ sufficiently large, induces a pseudoprojection on $H_*\Sigma X$. As indicated earlier, we then use the telescope construction to obtain a splitting

$$\Sigma X \simeq K \vee L$$

where the homology of $K$ realizes the image of $\psi^n_*$, and $H_* L$ realizes the image of $(1 - \psi^n)_*$. The rest of the proof proceeds very much like the proof of Proposition 3.2; we next split a finite dimensional retract off $L$, using a new self map of $\Sigma X$—one that is homotopic to the identity up through dimension $t$ and that induces the zero map in homology in degrees greater than some $\tau$. In the end, we take the direct limit of these splittings to obtain

$$\Sigma X \simeq \bigvee_{\alpha} K_\alpha$$

where each $K_\alpha$ is finite dimensional. To finish the proof of this theorem, we need only to verify the lemma that we used.

*Proof of Lemma* 3.3.1. Choose an integer $\tau$ large enough that the composition

$$\Sigma X \xrightarrow{j} \bigvee^{\infty} \Sigma X_n \xrightarrow{\pi_\tau} \bigvee_{1 \le n \le \tau} \Sigma X_n \xrightarrow{\mathcal{F}_\tau} \Sigma X_\tau$$

induces the identity on $H$. Here we have used the inclusion $\Sigma X_\tau \to \Sigma X$ to identify the homology of the two spaces in degrees $\le t$. Now $\Sigma X_\tau$ has the rational homotopy type of a finite bouquet of spheres; let $W$ denote the subbouquet consisting of those spheres of dimension $\le t$. Since $\Sigma X_\tau$ is a finite co-$H$-space, Wilkerson's Theorem 3.2 shows that there is an integer $r$, and maps

$$\Sigma X_\tau \longrightarrow W \longrightarrow W \longrightarrow \Sigma X_\tau$$

whose composition realizes the endomorphism $p^r A$ on $V$. So take the composition $\Sigma X \to \Sigma X_\tau$ mentioned first, follow it by this one, and then compose that by the inclusion $\Sigma X_\tau \to \Sigma X$. This is the required map. □

*Corollary* 3.5. This follows easily from Theorem 3.3. □

*Proof of Example* 3.6. If $G$ is a finite group, let $p$ be a prime that divides its order. If $G$ is not discrete, it will have a maximal torus and so in this case $p$ could denote any prime. Next let $N$ denote the ideal of nilpotent elements in $H^*(BG, \mathbb{Z}/p)$. Thus $u \in N$ if and only if $u^n = 0$ for some positive integer $n$. A theorem of Quillen asserts that Krull dimension of the commutative $\mathbb{Z}/p$ algebra $H^*(BG, \mathbb{Z}/p)/N$ equals the maximum rank of an elementary abelian $p$-subgroup of $G$, [20]. In particular, this dimension is at least one and so there is at least one class $u \in H^*(BG, \mathbb{Z}/p)$, of positive degree, with $u^n \neq 0$ for all positive integers $n$. It follows that the oribit of $u$ under the action of the Steenrod algebra is an infinite set. Thus $\widehat{BG}_p$ contains a stable retract that is infinite dimensional and atomic. The example now follows from Corollary 3.5. A final remark: if $G$ is a finite group, then the Segal conjecture—or rather its confirmation—shows that every proper stable retract of $BG$ is infinite dimensional. □

*Example* 3.7. Given that $\Omega G$ is stably atomic, the rest follows easily. □

*Proof of Example* 3.8. Let $G$ denote the localization at $p = 2$, of $SU(3)$. By way of a contradiction, assume by Theorem 2 that there is an inclusion,

$$j: \Sigma\Omega G \longrightarrow \bigvee^{\infty} \Sigma(\Omega G)_n$$

which is a right inverse to the folding map $\mathcal{F}$. Use these maps, along with the projections within the bouquet, to construct a map

$$\phi_t: \Sigma\Omega G \longrightarrow \Sigma(\Omega G)_t \longrightarrow \Sigma\Omega G$$

that induces a homology isomorphism in degree 3. We did this once before in the proof of Theorem 3.3. Now apply the loop functor to this map and consider the next composition

$$\Omega G \xrightarrow{E} \Omega\Sigma(\Omega G) \xrightarrow{\Omega\phi_t} \Omega\Sigma(\Omega G) \xrightarrow{r} \Omega G$$

where $E$, is the suspension map and the last map $r$ is a retraction—a left inverse for $E$. This composition induces a homology isomorphism in degree 2. It can be rewritten as follows

$$\Omega G \longrightarrow \Omega\Sigma((\Omega G)_t) \longrightarrow \Omega G.$$

Now $\Omega G$ is atomic, according to Hubbuck, [11], Theorem 1.1, and so this composition must be an equivalence. In particular, the first map induces a monomorphism in mod 2 homology—one that commutes with the the Steenrod algebra, acting on the homology of both spaces. However, the next result shows no such map exists.

PROPOSITION 3.8.1. *It is not possible, for any integer t, to embed* $H_*(\Omega G; \mathbf{Z}/2)$ *in* $H_*(\Omega\Sigma((\Omega G)_t); \mathbf{Z}/2)$, *as an unstable module over the Steenrod algebra.* □

There is a standard way to put an $\mathscr{A}_2$-module structure on the mod-2 homology of a space $X$ of finite type. One starts with the natural isomorphism,

$$H^q(X; \mathbf{Z}/2) \xrightarrow{h} Hom(H_q(X; \mathbf{Z}/2), \mathbf{Z}/2)$$

from the universal coefficient theorem, [22], then applies $Hom( , \mathbf{Z}/2)$ to both sides, and then invokes the natural isomorphism between $V$ and $V^{**}$ when $V$ is a finite dimensional vector space. This has been done in the proof that follows, although the notation will not reflect this. We will use $Sq^i$ to denote that which might more precisely be described as $(Sq^i)^*$ or $Hom(Sq^i, \mathbf{Z}/2)$.

To simplify notation further, let $H$ denote $H_*(\Omega SU(3); \mathbf{Z}/2)$. Then, as an algebra over the Steenrod algebra,

$$H \approx \mathbf{Z}/2[x, y]$$

where $x$ has degree 2, and $y$ has degree 4, and $Sq^2(y) = x$. This and the Cartan formula completely determine the action of the Steenrod algebra on $H$. Let $T(k)$ denote $H_*(\Omega\Sigma((\Omega G)_k); \mathbf{Z}/2)$; it is isomorphic to the tensor algebra,

$$T(k) \approx T(\bar{H}_*((\Omega G)_k; \mathbf{Z}/2).$$

The action of $\mathscr{A}_2$ on $T_k$ is completely determined by its action on the $k$-skeleton of $H$ and the Cartan formula. Given a monomial $z \in H$, of degree $\le k$, we will let $[z]$ denote the corresponding algebra generator in $T(k)$. Thus a typical element in $T(k)$ may be expressed as a sum of monomials of the form

$$[x^{a_1} y^{b_1}][x^{a_2} y^{b_2}] \ldots [x^{a_q} y^{b_q}]$$

It suffices to show that there is no embedding of $H$ into $T(2^n)$, as unstable modules over $\mathscr{A}_2$, for any integer $n$. By way of a contradiction, assume that such an embedding does exist; take $n$ to be minimal and let $N = 2^{(n-1)}$. There is an important relation in $H$,

$$Sq^k(y^N) = \begin{cases} x^N & \text{if} \quad k = 2N \\ 0 & \text{if} \quad k \neq 2N \quad \text{and } k > 0 \end{cases}$$

which can be verified using the Cartan formula and induction on $n$. This relation implies that an embedding of $H$ into $T(2^n)$ must take $y^N$ to a class, say $z$, for which $Sq^{2N}(z) \neq 0$ while $Sq^k(z) = 0$, for all positive $k < 2N$. We will show that there is no such class $z$, of degree $4N$, with these properties in $T(2^n)$.

We first claim that if $z$ is a monomial of degree $4N$ in $T(2^n)$, then $Sq^{2N}(z) = 0$ if and only if $x$ divides $z$ (more precisely, iff some $[x^a y^b]$ divides $z$, where $a > 0$). To prove this claim, suppose first that $x$ does not divide $z$. Then $z$ has the form,

$$z = [y^{k_1}][y^{k_2}] \ldots [y^{k_q}] \text{ where } k_1 + k_2 + \ldots + k_q = N.$$

The Cartan formula and the relation just mentioned implies that

$$Sq^{2N}(z) = [x^{k_1}][x^{k_2}] \ldots [x^{k_q}]$$

which is certainly nonzero. Next suppose that $z$ is a divisible by $x$; say

$$z = [x^a y^b]u, \text{ where } a > 0, \text{ and } b \geq 0.$$

The largest $k$ for which $Sq^k([x^a y^b]) \neq 0$ is $2b$. But since $4N - 4b > 4N - (2a + 4b) =$ degree $(u)$ it follows that $Sq^{2N-2b}(u) = 0$, by the unstable axiom, and hence $Sq^{2N}(z) = 0$, by the Cartan formula. This case was admittedly special, but it is clear that the general case follows for the same simple reasons. Our claim thus established, it follows that an embedding of $H$ into $T(2^n)$ must take $y^N$ to a nonzero linear combination of monomials in $[y]$, $[y^2], \ldots, [y^{N/2}]$, plus possibly, an "error term" $u$ on which $Sq^{2N}(u) = 0$.

LEMMA 3.8.2. *Let $S(4N)$ denote the set of nonzero monomials in $[y], [y^2], \ldots, [y^{N/2}]$, of degree $4N$, in $T(2^n)$. Then,*

(a) *for each monomial $z$ in $S(4N)$, there is a $k$ between 1 and $N/2$ such that $Sq^{2k}(z) \neq 0$.*
(b) *Suppose $z = \Sigma z_i$ where each $z_i \in S(4N)$ and each $Sq^{2k}(z_i) \neq 0$. Then $Sq^{2k}(z) = 0 \Rightarrow z = 0$.*
(c) *Assume $z = \Sigma z_i$ where each $z_i \in S(4N)$ and that $u$ is the error term mentioned earlier. Then $Sq^{2k}(z + u) = 0 \Rightarrow Sq^{2i}(z) = 0$.*

It is clear that this lemma yields the contradiction we seek. Combined with the claim established earlier, it shows that the Steenrod algebra actions on $H$, and on $T(2^n)$, are sufficiently different that no embedding, over $\mathscr{A}_2$, of the first into the second is possible. In retrospect, this is perhaps not surprising, in view of the commutativity in $H$, which allows many terms to cancel out, and the noncommutative nature of $T(2^n)$, which prevents this.

*Proof of* 3.8.2. For part (a), take a typical monomial, say

$$z = [y^{k_1}][y^{k_2}] \ldots [y^{k_q}]$$

in $S(4N)$ and let $k = \max\{k_1, k_2, \ldots, k_q\}$. Then

$$Sq^k(z) = \sum_{k_i = k} [y^{k_1}] \ldots [x^{k_i}] \ldots [y^{k_q}] + \text{other terms}$$

These other terms could be nonzero only if $k$ equals a sum of smaller $k_i$'s. In this case there will be two or more of the $[x^{k_i}]$-factors appearing in each of the "other terms". They will be linearly independent of those displayed above.

For the proof of part (b), notice that there is a simple method for recovering a $z_i \in S(4N)$ from a nonzero $Sq^{2k}(z_i)$: when the latter is expressed as a sum of monomials in the $[x^a]$'s and $[y^b]$'s, simply take any one of the monomials and replace each occurrence of $x$ in it by $y$. Part (b) follows easily from this observation.

In part (c), recall that $x^a$, for some $a \geq 1$, divides the error term, $u$, whereas $x$ does not divide any member of $S$(4N). It follows then, that $k$ is the highest power of $x$ that divides $Sq^{2k}(z)$ whereas $x^{k+a}$ divides $Sq^{2k}(u)$. The two values of $Sq^{2k}$ must then be linearly independent in $T(2^n)$ and so part (c) follows. □

*Proof of Example* 3.9. At the prime 2, the space $X$ is equivalent to a bouquet of spheres, while at an odd prime $l$, $X$ is equivalent to

$$\left(\bigvee_{p \neq l} S^{2p+1}\right) \vee \left(S^3 \cup_{\alpha_1} e^{2l+1}\right)$$

Hence the universal phantom map out of $X$ is trivial at each prime by Theorem 3.3.

Suspend once and consider phantom maps into $S^5$ using the isomorphism of abelian groups,

$$Ph(\Sigma X, S^5) \approx \varprojlim{}^1 [\Sigma^2 X, (S^5)^{(n)}].$$

Let $A_n = [\Sigma^2 X, (S^5)^{(n)}]$. Notice that $A_5 = \mathbf{Z}$ and that for larger $n$, $A_n \approx \mathbf{Z} \oplus F_n$ where $F_n$ is a finite abelian group. This follows by rationalizing the spaces and groups involved. The quotient map $A_n \to A_n$/torsion, is known to induce a $\varprojlim^1$ isomorphism, [13], and so we can disregard the torsion summand. Any map

$$S^5 \bigcup_{\alpha_1} e^{2l+3} \longrightarrow S^5$$

must induce in $H_5(\ ; \mathbf{Z})$ a homomorphism whose degree is divisible by $l$. It follows that the image of $A_n$ in $A_5$ has finite index divisible by all primes $l$ with $2l + 3 \leq n$. The tower $\{A_n\}$ is therefore not Mittag-Leffler and so, by Theorem 2 of [16], its $\varprojlim^1$ term is nonzero. Hence essential phantom maps exist between $\Sigma X$ and $S^5$. Obviously this argument adapts to higher suspensions of $X$ with the same conclusion. The stable nontriviality of $\Theta$ follows easily. □

*Proof of Example* 4.1. We are going to prove something slightly stronger than the result stated. Let $\Theta(X, Y)$ denote the set of pointed homotopy classes of maps from $X$ to $Y$ whose restriction to each finite subcomplex of $X$ is null homotopic. It is clear that $Ph(X, Y) \subseteq \Theta(X, Y)$. Of course, the two sets are equal when $X$ has finite type, but in general, they are not equal, [7]. We will show that $\Theta(X, Y) = 0$ when $X$ and $Y$ satisfy the conditions of Example 4.1. Our proof requires three lemmas. The first one is a general fact and some special cases of it are implicit in the literature.

LEMMA 4.1.1. *Let $X$ be a pointed CW-complex and let $Y$ be a pointed space. Let $\{X_\alpha\}$ denote the directed system of all pointed finite subcomplexes of $X$. Then, as pointed sets,*

$$\varprojlim{}^1 [\Sigma X_\alpha, Y] \approx \Theta(X, Y) \qquad \square$$

When $X$ has finite type, the direct system mentioned here can be replaced by the cofinal sequence of skeleta and one can appeal to [2], page 256, for this result. When the target $Y$ is nilpotent and of finite type this first lemma is a consequence of [23], Theorem 3.3.

Recall that an inverse system of groups $\{G_\alpha\}$ over a directed set, is said to satisfy the Mittag-Leffler condition if for each $\alpha$, there is a $\beta > \alpha$, such that

$$image\{G_\alpha \leftarrow G_\beta\} = image\{G_\alpha \leftarrow G_\gamma\}$$

for all $\gamma \geq \beta$. The next result is well known and well documented in the special case of inverse *sequences* of groups, e.g., [2], [15]. It is probably well known in the generality we require—for groups which are not necessarily abelian, indexed by directed sets which are not necessarily countable—but we were unable to find it in the literature.

LEMMA 4.1.2. *Let $\{G_\alpha\}$ be an inverse system of groups, indexed by a directed set. If this system satisfies the Mittag-Leffler condition, then $\varprojlim^1 G_\alpha = *$.* □

LEMMA 4.1.3. *Let $X$ and $Y$ satisfy the conditions of Example 4.1. For each finite subcomplex $X_\alpha$, of $X$, there is another finite subcomplex, $X_\beta$, which contains it and for which the image of the restriction map,*

$$[\Sigma X_\alpha, Y] \xleftarrow{\ i^*_{\alpha\beta}\ } [\Sigma X_\beta, Y]$$

*is a finite subgroup.* □

This third lemma clearly implies that the system $\{[\Sigma X_\alpha, Y]\}$ is Mittag-Leffler and so Example 4.1 follows from the previous two lemmas. We now give their proofs.

*Proof of* 4.1.1. First, a note about basepoints: all maps considered in this proof preserve basepoints and all homotopies fix these basepoints. The cones, suspensions, and similar constructions we use here will be reduced, and thus will also have basepoints.

Given a phantom map, $f: X \to Y$, choose for each finite subcomplex $X_\alpha$, a null homotopy of the restriction $f|X_\alpha$ and use it to define an extension of $f|X_\alpha$ to the cone $\hat{f}_\alpha: CX_\alpha \to Y$. If $X_\beta$ is another finite subcomplex that contains $X_\alpha$, define a map

$$F_{\alpha\beta}: \Sigma X_\alpha \longrightarrow Y \text{ to be } \begin{cases} \hat{f}_\beta | CX_\alpha & \text{on the upper cone} \\ \hat{f}_\alpha & \text{on the lower cone} \end{cases}$$

Of course, this collection of maps $\{F_{\alpha\beta}\}$ is not uniquely determined by $f$; it depends on a choice, for each $\alpha$, of null homotopy used in defining $\hat{f}_\alpha$. These choices, up to homotopy, are in one to one correspondence with $[\Sigma X_\alpha, Y]$. Fortunately, this problem disappears in $\varprojlim^1 [\Sigma X_\alpha, Y]$. Recall how it is defined: for each $\alpha \leq \beta$, let $G_{\alpha\beta} = G_\alpha = [\Sigma X_\alpha, Y]$ and let $i^*_{\alpha\beta}: G_\alpha \leftarrow G_\beta$ be induced by restriction. The set of cocycles, $Z(G_\alpha)$, is then defined as

$$Z(G_\alpha) = \left\{ z_{\alpha\beta} \in \prod_{\alpha \leq \beta} G_{\alpha\beta} \,|\, z_{\alpha\beta} \cdot i^*_{\alpha\beta}(z_{\beta\gamma}) = z_{\alpha\gamma}, \text{ whenever } \alpha \leq \beta \leq \gamma \right\}$$

The maps $F_{\alpha\beta}$ are easily seen to satisfy this cocycle condition. Now define a right action

$$Z(G_\alpha) \times \prod_\alpha G_\alpha \longrightarrow Z(G_\alpha)$$

by

$$(z_{\alpha\beta}) \cdot (g_\gamma) = (g_\alpha^{-1} \cdot z_{\alpha\beta} \cdot i^*_{\alpha\beta}(g_\beta))$$

The orbit space of this action is $\varprojlim^1 G_\alpha$. From this description, it follows that the correspondence

$$\Theta(X, Y) \xrightarrow{f \mapsto (F_{\alpha\beta})} \varprojlim^1 [\Sigma X_\alpha, Y]$$

is well defined. Indeed, if one uses different extensions, say $\{\hat{f}_\alpha + g_\alpha\}$ to create a second family, $\{F'_{\alpha\beta}\}$, then it follows that $(F_{\alpha\beta}) \cdot (g_\alpha) = (F'_{\alpha\beta})$. To obtain an inverse correspondence, we will use a slightly different universal phantom map out of $X$,

$$X \xrightarrow{\Theta} \bigvee_{\alpha < \beta} \Sigma X_{\alpha\beta},$$

where $X_{\alpha\beta} = X_\alpha$. To define this, we first replace the telescope of skeletons, $Tel(X)$, by a similar construction, $\mathcal{T}(X)$, which is also homotopy equivalent to $X$. From a distance, $\mathcal{T}(X)$ resembles a tree; up close, one sees that each branch of it is a telescope. To be more precise, we will construct $\mathcal{T}(X)$ as a direct limit,

$$\mathcal{T}(X) = \varinjlim T(X_\alpha)$$

where $X_\alpha$ runs through all finite subcomplexes of $X$. First we describe $T(K)$ where $K$ is a finite complex. Let $K_1, K_2, \ldots, K_n$ be the list of all proper subcomplexes of $K$ with those that are maximal in $K$ listed before those which are not. The first stage in the construction of $T(K)$ is to take the mapping cylinder of each inclusion, $K_i \to K$, and then glue all of them together along $K$. In other words, take

$$\left( \coprod_{1 \leq i \leq n} K_i \times I \right) \coprod K$$

and in each $K_i \times I$, identify the points $(x, 1)$ with $x$ in $K$. Call the result $T_1(K)$. The next step is to repeat this process on just the maximal subcomplexes, say $K_1$ and $K_2$, of $K$. Having done this, identify the $t = 1$ end of $T_1(K_1)$ with $K_1 \times \{0\}$ in $T_1(K)$. Likewise, the $t = 1$ end of $T_1(K_2)$ gets identified with $K_2 \times \{0\}$ in $T_1(K)$. This completes the construction of the second stage, $T_2(K)$. The third stage of the construction involves only the maximal subcomplexes of $K_1$ and $K_2$, to which this process would be repeated. Continue in this manner as many times as are necessary; call the end result $T(K)$. This construction requires as many stages as the maximum length of any chain in the lattice of subcomplexes of $K$. The similarity between $T(K)$ and a tree should be clear. Each branch corresponds to a strictly decreasing sequence of subcomplexes that starts with $K$; each successor, with the possible exception of the last one, is maximal in its immediate predecessor. This correspondence is bijective. So, if $L$ is a subcomplex of $K$, it is clear that $T(L)$ can be identified with a subtree of $T(K)$, but not necessarily in a unique way ($L$ could be maximal in more than one complex). We are forced to make choices. For each finite subcomplex $K$ of $X$, choose another finite subcomplex $K'$, in which it is maximal. Do this in such a way that the choices compose to form a directed system of inclusions. To put it another way, choose a maximal tree in the Hasse diagram of the lattice of finite subcomplexes of $X$. Then use the induced inclusion of $T(L)$ in $T(K)$, whenever $L < K$, to construct $\mathscr{T}(X)$ as the topological direct limit of the $T(X_\alpha)$'s. There is an obvious map $\mathscr{T}(X) \to X$. It is the direct limit of homotopy equivalences $T(K) \to K$. As such, it is easily seen to be a homotopy equivalence.

The universal phantom map is then defined as the quotient map from $\mathscr{T}(X)$ to the bouquet $\bigvee_{\alpha<\beta} \Sigma X_{\alpha\beta}$, that sends to the basepoint all points in $\mathscr{T}(X)$ whose second coordinates were once integers. Moreover, for each $\alpha < \beta$, the mapping cylinders of $X_\alpha \subset X_\beta$ in $\mathscr{T}(X)$, of which there are many, get sent to the lone $\Sigma X_{\alpha\beta}$.

Given any family of maps $\{h_{\alpha\beta}: \Sigma X_\alpha \to Y\}_{\alpha<\beta}$ it is clear that by composing $\bigvee_{\alpha<\beta} h_{\alpha\beta}$ with $\Theta$ one gets a phantom map from $X$ to $Y$. Moreover, if the two families satisfy the cocycle condition and represent the same class in $\varprojlim^1 G_\alpha$, then composing them with $\Theta$ will yield the same phantom map.

Thus we have a function

$$\varprojlim{}^1 [\Sigma X, Y] \xrightarrow{\Theta^*} \Theta(X, Y)$$

We claim that $\Theta^*$ is an inverse to the function defined earlier. To see this, take a phantom map $f: X \to Y$ and for each $\alpha$ choose a null homotopy, $N_\alpha$ of $f | X_\alpha$. Then compose $f$ with the projection $\pi: \mathscr{T}(X) \to X$. Use the $N_\alpha$'s to construct a homotopy, $H_s$, from $f\pi$ to a map that factors through the bouquet $\bigvee_{\alpha<\beta} \Sigma X_{\alpha\beta}$. On a typical mapping cylinder of $X_\alpha \subset X_\beta$ in $\mathscr{T}(X)$, this homotopy is given by the formula

$$H_s(x, t) = \begin{cases} N_\alpha(x, s(1 - 2t)) & \text{if } 0 \le t \le 1/2 \\ N_\beta(x, s(2t - 1)) & \text{if } 1/2 \le t \le 1 \end{cases}$$

The $t = 0$ end of this homotopy is $f(x)$, while the $t = 1$ end has the form $\Theta^*\left(\bigvee_{\alpha<\beta} F_{\alpha\beta}\right)$. The claim follows. Composing these two functions in the opposite order induces the identity on $\varprojlim^1 [\Sigma X_\alpha, Y]$—as the reader can easily verify. □

*Proof of* 4.1.2. Notice that $\varprojlim^1 G_\alpha = *$ if and only if for each cocycle family $\{z_{\alpha\beta}\}$, there exists a family $\{g_\gamma\} \in \prod G_\gamma$ such that

$$g_\alpha = z_{\alpha\beta} \cdot f_{\alpha\beta}(g_\beta)$$

for each $\alpha < \beta$. If all the structure maps $f_{\alpha\beta}$ are surjective, then one can obtain such $g_\gamma$'s. First use this surjectivity and Zorn's lemma to define the $g_\gamma$'s for a maximal chain of $\gamma$'s. Then for any $\alpha$ off the chain choose a larger $\beta$ on the chain and define let $g_\alpha = f_{\alpha\beta}(g_\beta)$. It is straightforward to check that this works.

If all the $f_{\alpha\beta}$'s are surjective then, obviously, the system $\{G_\alpha\}$ satisfies the Mittag–Leffler condition. Conversely if $\{G_\alpha\}$ satisfies the Mittag–Leffler condition, then in the category of pro-groups, it is isomorphic to another progroup $\{G'_\alpha\}$ whose structure maps $\{f'_{\alpha\beta}\}$ are all epimorphisms, [15], page 167. Since $\varprojlim^1$ is a functor from the category of pro-groups to that of sets, *ibid.* p. 177, the lemma follows.

*Proof of* 4.1.3. Given $X_\alpha$, we will choose $X_\beta$ to be a slightly bigger, but still finite subcomplex of $X$, such that the image of

$$H_n(X_\alpha; \mathbf{Z}) \longrightarrow H_n(X_\beta; \mathbf{Z})$$

is finite for each $n \geq 1$. Of course, if $H_n(X_\alpha; \mathbf{Z})$ is finite for all $n > 0$, we will take $X_\beta = X_\alpha$. Otherwise, let $d$ denote the highest dimension in which the integral homology of $X_\alpha$ fails to be finite. Consider the long exact sequence for the pair $(X, X_\alpha)$,

$$\longrightarrow H_{d+1}(X, X_\alpha) \longrightarrow H_d X_\alpha \xrightarrow{i_*} H_d X \longrightarrow$$

The kernel of $i_*$ is finitely generated and the relative group is generated by the cells in $X \to X_\alpha$ of dimension $d+1$. Therefore, take a minimal collection of $(d+1)$-cells that, in homology, map onto the kernel of $i_*$, and let $K^d$ be the smallest subcomplex that contains them and $X_\alpha$. In homology, the image induced by the inclusion $X_\alpha \to K^d$ is evidently finite in degrees $\leq d$. Now consider $H_{d-1}K^d$; if it is not finite, kill off the kernel of $H_{d-1}K^d \to H_{d-1}X$, thereby creating $K^{d-1}$. Continuing in this manner, we get a sequence of inclusions

$$X_\alpha \longrightarrow K_d \longrightarrow K_{d-1} \longrightarrow \dots \longrightarrow K_1 = X_\beta$$

such that for each degree $n \geq 1$, one of these maps (and hence the composition of all of them) induces a finite image in $H_n$.

In cohomology with coefficients in a finitely generated abelian group, $A$, one sees that the restriction

$$H^n(X_\alpha; A) \xleftarrow{i^*} H^n(X_\beta; A)$$

also has a finite image. If $\Omega Y^{(n)}$ denotes the $n$-th stage of the Postnikov system for $\Omega Y$, it follows that the image of

$$[X_\alpha, \Omega Y^{(n)}] \xleftarrow{i^*} [X_\beta, \Omega Y^{(n)}]$$

is likewise finite. Here one uses the principal fibrations

$$K(\pi, n) \longrightarrow \Omega Y^{(n)} \longrightarrow \Omega Y^{(n-1)},$$

induction on $n$, and the finite type hypothesis on $Y$ to verify this. Finally, by taking $n$ sufficiently large one has the first of two isomorphisms

$$[X_\beta, \Omega Y^{(n)}] \approx [X_\beta, \Omega Y] \approx [\Sigma X_\beta, Y]$$

and the result follows. □

*Proof of Example* 4.2. Each member of Toda's $\alpha$-family on $S^3$ has order $p$, is stably essential, and is stably indecomposable. It follows that the mapping cone, $X$, is stably irreducible at $p$. Thus by Theorem 3.3, the universal phantom map of $X$ is essential—stably essential, in fact.

It is a simple exercise to construct maps

$$X \xrightarrow{f} \bigvee_{t \geq 0} S^{2t(p-1)+3} \xrightarrow{g} X$$

where the map $f$ has degree 1 on the bottom 3-sphere, and degree $p$ on each of the higher cells, and the composite $gf$, induces multiplication by $p$ on the reduced homology of $X$. The rest, of course, follows from Theorem 4.3, which is proved next. □

*Proof of Theorem* 4.3. Let $Y$ be a target of finite type and let

$$G_n = [X, \Omega Y^{(n)}]$$

We will identify $Ph(X, Y)$ with $\varprojlim^1 G_n$. The goal then, is to show that the tower $\{G_n\}$ is Mittag-Leffler. The following two properties of this tower will be used in the proof:

(i) each $G_n$ is a finitely generated nilpotent group, and
(ii) the image of $G_{n+1}$ in $G_n$ has finite index for each $n$.

This first property follows since both $X$ and $Y$ have finite type, [24], Chapter 10. The second property can be verified by applying the functor $[X, \ ]$ to the principal fibration

$$\Omega X^{(n+1)} \xrightarrow{\Omega p} \Omega X^{(n)} \xrightarrow{k} K(\pi, n)$$

Since the $k$-invariant is rationally trivial, the image of $k_*$ must be finite, and so its kernel must have finite index. The second property then follows by exactness.

Each $G_n$ is a countable group by (i). According to Lemma 3.2 of [16], a tower of countable groups satisfying property (ii) is Mittag-Leffler if and only if the canonical map $\varprojlim G_k \to G_n$ has a finite cokernel (i.e., its image has finite index in the target) for each $n$. There is an epimorphism

$$[X, \Omega Y] \longrightarrow \varprojlim G_k$$

by [23], Theorem 3.3a, and thus our tower is Mittag-Leffler if and only if the restriction map $[X, \Omega Y] \to G_n$ has a finite cokernel for each $n$. Of course, the same remarks apply to the tower $\{[Z, \Omega Y^{(n)}]\}$. With this in mind, consider the commutative diagram

$$\begin{array}{ccc} [X, \Omega Y] & \xleftarrow{f^*} & [Y, \Omega Y] \\ \downarrow & & \downarrow \\ [X, \Omega Y^{(n)}] & \xleftarrow{f^*} & [Z, \Omega Y^{(n)}] \end{array}$$

It is induced by restriction in the vertical direction and by the map $f: X \to Z$, in the horizontal direction. The assumption is that $Ph(Z, Y) = 0$. Hence the cokernel of the right side is finite. Assume for the moment, that the same is true of the map along the bottom. Going around the other way, it follows that the left side must also have a finite cokernel. We conclude, by earlier remarks, that $Ph(X, Y) = 0$.

To finish the proof, we need to show that the homomorphism along the base has a finite cokernel. Since the map $f$ is a rational homology equivalence, it follows that it also induces an isomorphism on the rationalization of the bottom edge. Therefore it suffices to establish the following.

LEMMA 4.3.1. *Let $\varphi: G \to H$ be a homomorphism between two finitely generated nilpotent groups. If the rationalization, $\varphi_0: G_0 \to H_0$, is an isomorphism, then the image $\varphi(G)$ has finite index in $H$.*

*Proof.* When the groups involved are abelian, this basic. For the general case, use the lower central series and the commutative diagram

$$\begin{array}{ccccc} \Gamma^i G & \longrightarrow & G & \longrightarrow & G/\Gamma^i G \\ \downarrow & & \varphi \downarrow & & \downarrow \\ \Gamma^i H & \longrightarrow & H & \longrightarrow & H/\Gamma^i H \end{array}$$

All three vertical maps in this diagram are rational equivalences if the middle one is, by [9], Theorems 6.5 and 6.6. Since the index of $\varphi$ equals the product of the indexes of the other two verticals, the result follows by induction on the maximum of nilpotency classes; $nil(G)$ and $nil(H)$. □

*Corollary* 4.4. This is a direct consequence of the theorem. □

*Proof of Corollary* 4.5. Assume that $K$ is not rationally trivial. Being a finite $H_0$-space, it follows that it must have the rational homotopy type of a finite product of odd dimensional spheres, say $P$. One can find maps going both ways that induce this rational equivalence,

$$K \xrightarrow{\varphi} P \xrightarrow{\gamma} K.$$

Here the maps $\varphi$ and $\gamma$ can be taken to be rational equivalences such that $\varphi\gamma$ and $\gamma\varphi$ both induce multiplication by some nonzero integer $\lambda$, on the rational homotopy groups of $K$. For a proof of this, see [16], Proposition 5.1. Thus the map $f$ in Theorem 4.3 can be taken to be a loop map in this special case. The corollary follows. □

*Corollary* 4.6. This is an immediate consequence of the previous corollary and the cited work of Roitberg. □

*Proof of Example* 4.7. For each prime $p$, and each $n \geq 1$, there is a rational equivalence

$$\Omega^2 S^{2n+1} \xrightarrow{\pi_p} S^{2n-1}_{(p)}$$

For $n \geq 2$, the degree of this map on the bottom cell must be at least $p$. These facts are some deep results of Cohen, Moore, and Neisendorfer. For a proof of these assertions see Cohen [4]-p. 558, for $p = 2$, see Neisendorfer [19]-p. 71, for $p = 3$, and see [5] for $p \geq 5$. Going in the other direction, there is, of course, the double suspension $E^2: S^{2n-1} \to \Omega^2 S^{2n+1}$, which exists without localizing and is a rational equivalence. It then follows, by Theorem 4.3, that

$$Ph(\Omega^2 S^{2n+1}, Y_{(p)}) = 0$$

for every $Y_{(p)}$ with finite type over $\mathbf{Z}_{(p)}$.

The existence of an essential phantom map from $\Omega^2 S^5$ to $\mathbf{HP}^\infty$ is a consequence of the loop space structure on $S^3$ as well as the nonexistence of global representatives for the maps

$\pi_p$, above. Notice that

$$Ph(\Omega^2 S^5, \mathbf{HP}^\infty) \approx \varprojlim{}^1 [\Omega^2 S^5, (\Omega \mathbf{HP}^\infty)^{(n)}] \approx \varprojlim{}^1 [\Omega^2 S^5, (S^3)^{(n)}]$$

Now for each $n \geq 3$, let

$$G_n = [\Omega^2 S^5, (S^3)^{(n)}]$$

As in 3.9, a rational calculation shows $G_n \approx \mathbf{Z} \oplus T_n$ where $T_n$ is a finite abelian group. The results of Cohen, Moore and Neisendorfer, imply that modulo torsion, the index of the image of $G_{n+k}$ in $G_n$ becomes divisible by more and more primes as $k$ increases. The assertion that $\varprojlim{}^1 G_n \neq 0$ then follows, just as it did in Example 3.9. □

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