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Vector Fields. Winding Number and Index. Poincare Theorem.

Vector Fields

A **vector field** V on a domain D of the plane is a function $V(P)$ associating to every point $P \in D$ a vector $V(P)$. Intuitively one can think of V as the velocity of some substance (for example, water) moving inside D. Placement of the vector $V(P)$ with its tail at P helps to visualize the vector field. For practical purposes it is usually more convenient to place all vectors with their tails at the origin. Then $V(P)$ may be described by the coordinates of its head:

$$V(P) = (F(x,y), G(x,y)), \tag{1}$$

where F and G are functions of $P = (x,y)$. The vector field V is called **continuous** when the functions F and G in (1) are continuous.

Vector fields have many important applications. The force fields arising from gravitation and electromagnetism are vector fields; the velocity vectors of a fluid motion, such as the atmosphere (wind vectors), form a vector field; and gradients, such as the pressure gradient on a weather map or the height gradient of a relief chart, are vector fields.

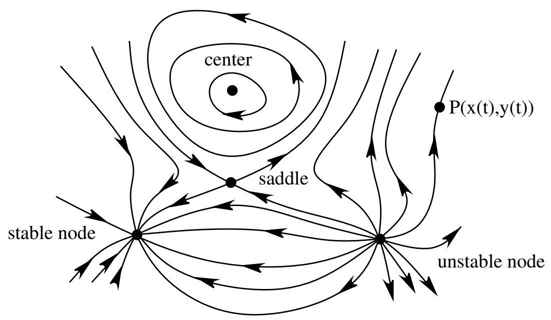
Vector fields are closely connected with differential equations. A vector field (1) can be interpreted as a system of differential equations

$$\begin{cases} dx/dt &= F(x,y) \\ dy/dt &= G(x,y) \end{cases}$$

A solution $(x(t), y(t))$ can be considered as the parametric representation of a directed path in the plane. The original vector field $V(P)$ gives the tangent vectors to the path at the point $P = (x,y)$. These directed paths with their tangent vectors defined by V form a **phase portrait** of the vector field. The paths are called **integral paths**, **trajectories**, **orbits**, or **flows**. The picture formed by the paths is called the **phase portrait** of the system of differential equations.

Critical points. It turns out that topological organization of the phase portrait is determined by the exceptional points P, called **critical points**, where $V(P) = 0$.

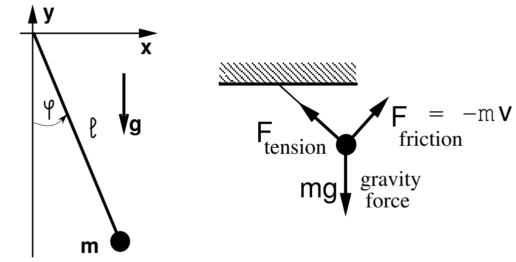
At the figure below there are four critical points. At the top there is a **center** characterized by the closed trajectories that swirl around it. No path passes through the center. At the bottom there are two critical points called **nodes** characterized by the fact that all the integral paths near these points end there. The difference between the two nodes can be expressed by saying that one is stable and the other one is unstable. In general, a critical point is called **stable** if one can find a cell surrounding the point from which no integral paths exit. The center is also a stable critical point. In between the center and the nodes there is a **saddle** where exactly four integral paths meet, two beginning and two ending. The saddle is unstable.



The most important characteristics of the phase portrait are the number and arrangement of the critical points, the pattern of the integral paths about each point, and stability or instability of the critical points.

Example: Simple Pendulum with Friction

Consider a simple pendulum oscillating in air. The frictional force is due to air resistance. It is directed in the opposite direction of the velocity of the pendulum $\mathbf{v}$, proportional to the absolute value of the velocity and given by $-k\mathbf{v}$.



Motion of the pendulum with friction is described by the following second-order differential equation

$$ml\frac{d^2\varphi}{dt^2} = -mg\sin\varphi - kl\frac{d\varphi}{dt}, \tag{2}$$

where m is the mass, l is the length of the pendulum, g is the gravitation acceleration.

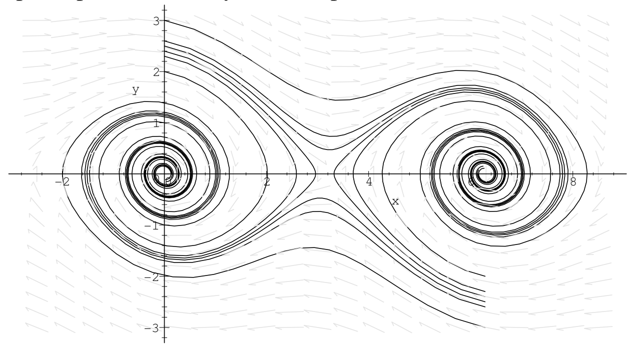
For certain m, l, and k equation (2) takes the form

$$\frac{d^2\varphi}{dt^2} = -\sin\varphi - 0.2\frac{d\varphi}{dt}$$

Denote $\varphi(t)$ by $x(t)$ and $d\varphi(t)/dt$ by $y(t)$. Then we arrive at the following system of two first-order differential equations:

$$\left.\begin{aligned} dx/dt &= y \\ dy/dt &= -\sin x - 0.2y \end{aligned}\right\} \tag{3}$$

The phase portrait of the system is exposed below.



Winding Number and Index

Consider a continuous vector field V and a closed curve γ. Suppose that there are no critical points of V on γ. Let us move a point P along the curve in the counterclockwise direction. The vector $V(P)$ will rotate during the motion. When P returns to its starting place after one revolution along the curve, $V(P)$ also returns to its original position. During the journey $V(P)$ will make some whole number of revolutions. Counting these revolutions positively if they are counterclockwise, negatively if they are clockwise, the resulting algebraic sum of the number of revolutions is called the **winding number** of V on γ.

Theorem 1 *Under a continuous deformation of a closed curve and a vector field the winding number does not change as long as the curve does not pass through a critical point of the field.*

Indeed, the direction of a vector of the filed varies continuously outside the critical points. Therefore the number of revolutions also varies continuously with the curve or with a continuous deformation of the field. Being an integer, it must be constant.

Theorem 2 *If the winding number of a curve is nonzero, then there is at least one critical point inside the domain bounded by the curve.*

The **index of an isolated critical point** of a vector field is the winding number of a small counterclockwise oriented circle with center at that point.

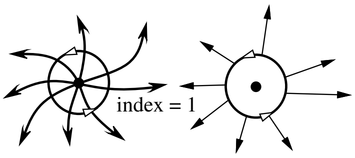


Figure 1: Index of a critical point.

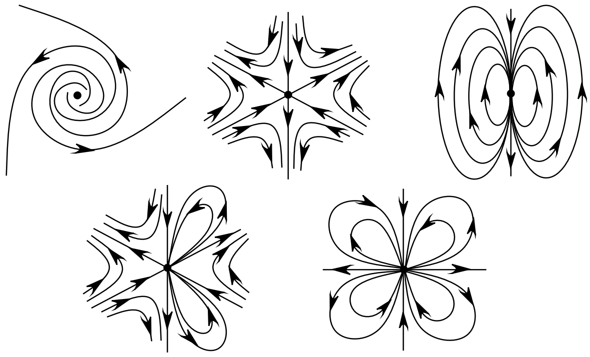


Figure 2: Critical points. Compute their indices.

Theorem 3 *Consider a curve and a vector field having no critical points on the curve. The winding number of the curve equals the sum of the indices of the critical points lying inside the domain bounded by the curve.*

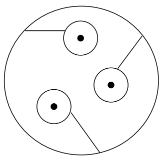


Figure 3: Proof of Theorem 3.

Theorem 4 *The winding number of a closed integral path of a vector field equals one.*

Tangent Vector Fields on Surfaces

Consider a smooth closed surface. Let each point of the surface be equipped with a vector tangent to the surface at that point. If the vectors depend continuously on the points where their tails are attached, we will say about (continuous) **tangent vector field** on the surface. Note that the definitions of critical points, winding numbers and indices can be trivially extended on for a tangent vector filed on a smooth surface.

Theorem 5 (Poincare) *Let V be a tangent vector field on a smooth closed oriented surface S with only isolated critical points P_1, P_2, ..., P_n. Then the sum of indices of the critical points equals the Euler characteristic of the surface:*

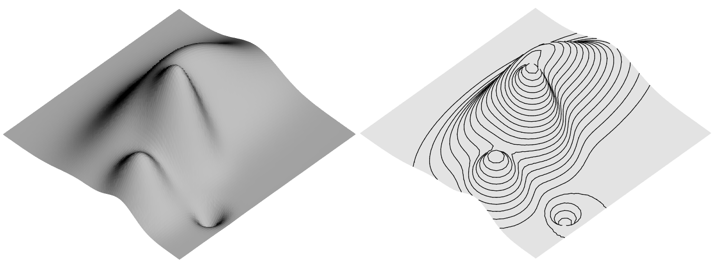
$$\sum_{i=1}^{n} \operatorname{ind}_V(P_i) = \chi(S).$$

Contours for Topography Visualizing

Imagine that a smooth function $z = f(x, y)$ is the height function of a mountainous island. The island topography can be usefully visualized in terms of the **contours**, the level sets

$$\{(x, y) \,:\, f(x, y) = \text{const}\}\,.$$

These contours (or in practice a suitable selection of them) convey information about the surface of the island.



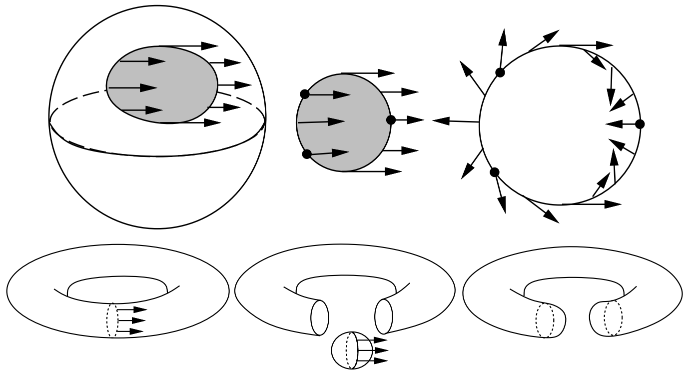


Figure 4: Top: proof of the Poincare theorem for a sphere. Bottom: proof of the Poincare theorem for a torus.

The contour lines are the integral paths for the vector field

$$V(x, y) = (\partial f/\partial y, -\partial f/\partial x)$$

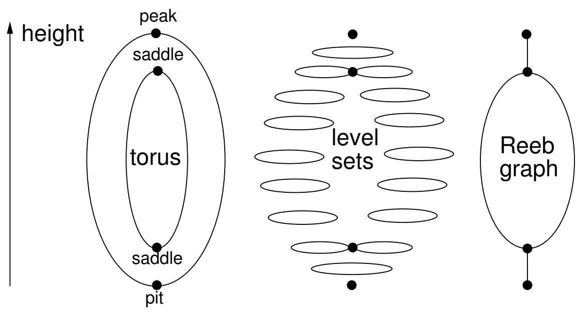
and, therefore, are the solutions of the system differential equations

$$\begin{cases} dx/dt &= \partial f/\partial y, \\ dy/dt &= -\partial f/\partial x. \end{cases}$$

Note that the critical points of $f(x, y)$ are the critical points of the system. Moreover, the maxima and minima are centers and the saddles are saddles.

A **topographical map** is a set of contours of a height function used for visualization purposes.

The Reeb graph. Consider a height function $z = f(x, y)$ on a closed surface. Let us construct a graph such that each closed contour is represented by a point in the graph. The contours merge, split, appear, or disappear at critical points of the height function. These bifurcation points are the nodes of the graph.



Problems

1. Draw a vector filed on the torus
   (a) without critical points
   (b) with two centers and two saddles
   (c) with one center and one saddle point
   (d) with a dipole and two saddle points
2. Draw a vector filed on the sphere with
   (a) two nodes
   (b) two centers
   (c) only one critical point
   (d) three critical points
3. Prove that at any time, there is a point on the earth where the wind is not blowing.
4. Consider an island.
   (a) Suppose the island has no lakes. Show that $\#\,piks - \#\,passes + \#\,pits = 1$.
   (b) Suppose now that the island has lakes. Prove that $\#\,piks - \#\,passes + \#\,pits = 1 - \#\,lakes$.